Algebraic Geometry

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Chapter 1

Introduction to Algebraic Geometry

Generally, people denote that Algebraic Geometry is the study of the zero set of polynomials. For example:

/images/circ1.png

Figure 1.1: Unit Circle in the Reals

But we will be looking at solutions in specifically $\mathbb{C}$.

1.1 Algebraic Curves of the Plane

Definition 1.1.1: Affine Space

Let $\mathbb{A}^n = \mathbb{C}^n$, which are n-tuples of complex numbers

For example we can have $(z_1, z_2) \in \mathbb{C}^2$ where $z_1, z_2 \in \mathbb{C}$. In fact, $\mathbb{C}[x_1, \ldots, x_n]$ is the polynomial algebra in n-variables.

If $f(x, y) = x^2 + y^2 - 1$, then $\{f = 0\} \subseteq \mathbb{C}^n = \mathbb{A}^n$, or $\{x^2 + y^2 - 1 = 0\}$

Definition 1.1.2: Affine Plane Curve

In fact $f(x_1, x_2) = x_1^2 + x_2^2 - 1$, thus $\{f = 0 : x_1, x_2 \in \mathbb{C}\} \subseteq \mathbb{A}^2$, the affine plane, called an affine plane curve. Note that this curve is a $\mathbb{R} - \dim(f) = 2$, but $\mathbb{C} - \dim(f) = 1$

We often only plot the real component of affine plane curves. But if $\mathbb{A} = \mathbb{C}$, it is called an affine line, where $\mathbb{C} - \dim = 1$ and $\mathbb{R} - \dim = 2$, since we can create a complex number with two real numbers adjoined with i.This is due to the fact that if $y - x^2 = 0$, then we cannot draw a complex graph, but we can draw the real component as a parabola.

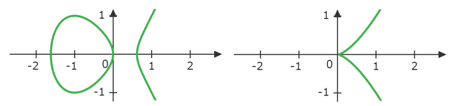


Figure 1.2: Two Graphs of Real Components

Definition 1.1.3: Affine Algebraic Curve

C is an affine curve defined by $\{f(x, y) = 0, x, y \in \mathbb{C}\} \subseteq \mathbb{A}^2$.

Definition 1.1.4: Degree of C

This is saying the same as the degree of f, which is the sum of the highest power in each variable.

$$\deg(\{xy + x + y = 0\}) = 2$$

. But all degree 2 affine plane curve polynomials are conics.

Lemma 1.1.1

Every affine conic is of the form:

1. $x^2 - y^2 = 1$ (Hyperbola);
2. $y = x^2$ (Parabola),

which are equivalent by change of coordinates (basis).

Example 1.1.1

Show that ellipses can change into one of these forms.

Throughout this course we will be working with algebraically closed fields, for example $\mathbb{R}$ is not algebraically closed.

Definition 1.1.5: Algebraically Closed

If every non-constant polynomial has a root in the field.

An example that $\mathbb{R}$ is not closed is shown when

$$\{x^2 + 1 = 0 : x \in \mathbb{R}\} = \phi$$

For any field, $\mathbb{F}$, there exists $\bar{\mathbb{F}}$ that is the smallest algebraically closed field containing $\mathbb{F}$ called the algebraic closure of $\mathbb{F}$. For example $\bar{\mathbb{R}} = \mathbb{C}$.

Proposition 1.1.1

Let $\mathbb{K}$ be a arbitrary field, $f, g \in \mathbb{K}[x, y]$ which are irreducible polynomials, meaning ti cannot be factored into non-constant. If g is not divisible by f, then $f(x, y) = g(x, y) = 0$ by only finitely many satisfying solutions.

We can easily understand the relationship between $\{f(x, y) = 0 : x, y \in \mathbb{C}\} \longrightarrow C$, but what about the inverse direction. This is one of our goals in Algebraic Geometry. Given C, an algebraic curve which is irreducible, then we can understand its defining $\{f(x, y) = 0\}$ up to scalar.

Definition 1.1.6: Smooth Curve

A polynomial equation that is differentiable at all points.

Lemma 1.1.2

Any smooth affine conic is isomorphic to an affine plane curve.

In fact, over $\mathbb{R}$ any conic can be given by either a hyperbola, parabola, or ellipse; However, this is not true over the complexes. If we have $C \cong C'$ as affine plane curves, then we can write the following commutative diagram:

$$\begin{array}{ccc} \text{C} & \hookrightarrow & \mathbb{A}^2 \\ \cong\downarrow & \circlearrowleft & \downarrow\cong \\ \text{C'} & \hookrightarrow & \mathbb{A}^2 \end{array} \quad \begin{array}{c}\text{Change in} \\ \text{Coordinates}\end{array}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} M \end{bmatrix}_{2x2} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} T \end{bmatrix}_{2x1}$$

Figure 1.3: Isomorphism of Affine Curves through Change in Basis

If we take the following conic $C : \{f(x, y) = x^2 + y^2 - 1 = 0\}$, which is a circle, then we can translate it to a conic in the complexes by:

$$\begin{aligned} x &\mapsto x \\ y &\mapsto iy \\ x^2 + y^2 - 1 &\mapsto x^2 - y^2 - 1 \end{aligned}$$

1.2 Rational Curves

Let C be an affine plane curve in $\mathbb{A}^2$.

Definition 1.2.1: Rational

C is rational if it can be parametrized by rational functions such that there exists rational functions φ, ψ such that

$$\begin{aligned} x &= \varphi(t) \\ y &= \psi(t), \end{aligned}$$

and $f(x, y) = 0$

Definition 1.2.2: Rational Function

A quotient of rational polynomials.

$$\frac{t^3 + 3t^2 + 5}{t - 1}$$

is a rational function. The domain is $\mathbb{C} \setminus \{1\}$. In fact, any complex line is rational due to parametrization.

$$f(x, y) = 2x + 3y - 5,$$

We can parameterize

$$x = t$$
$$y = \frac{-2x + 5}{3}$$

$$f(x, y) = x - 2,$$

$$x = 2$$
$$y = t$$

We can even parametrize the conic $C : \{y - x^2 = 0\}$ as

$$x = t$$
$$y = t^2$$

But simply allowing one to keep parametrizing these conics through a process of implicit isolation, we cannot have a rational function at all times. For example:

$$C : \{x^2 - y^2 = 1\}$$

$$x = t$$
$$y = \sqrt{t^2 - 1},$$

but the square root is not rational. Thus we must look at an isomorphism of this affine plane curve, by drawing a morphism between $C \cong C' : \{x^2 + y^2 - 1 = 0\}$

Note that any smooth conic is rational, due to being over any algebraically closed field.

Lemma 1.2.1

If C is rational plane curve, and $C \cong C'$, then C' is rational.

Lemma 1.2.2

Any affine smooth cubic that is also infinitely smooth is never rational. And any smooth projective cubic curve is never rational.

A counter example of this is if we have genus $\mathfrak{g} = \frac{(d-1)(d-2)}{2}$. Any $d = 4$ with 3 nodes contains all rational curves without a line to connect all other points.

1.3 Relation to Field Theory

Let $X := \{f(x, y) = 0 \subseteq \mathbb{A}^2$ over $\mathbb{C}\}$. And assume that X is irreducible.

Lemma 1.3.1

Consider a rational function $u(x,y) = p(x,y)/q(x,y)$ and $(f \nmid q)(x,y)$. Let $u(x,y) \cong u_1(x,y) = \frac{p_1(x,y)}{q_1(x,y)}$ if and only if $pq_1 - p_1q$ is divisible by f.

Note that $\{f = 0\} \cap \{q = 0\}$ is only finitely many points when $f \nmid q$. Thus we can simplify this lemma into a simple statement of

Lemma 1.3.2

$u(x,y) \cong v(x,y)$ if and only if $f|(pq_1 - p_1q)$

Definition 1.3.1: Function Field

Define $\mathbb{C}(X)$: field of all such rational functions.

But why do we want $f \nmid g$?

It is because when $u = p/q$, it is a rational function on $\mathbb{A}^2 \setminus \{q = 0\}$. Thus we can consider the notation $\frac{p}{q}\Big|_X$, which is the notation for the restriction on X. Which is if and only if $f \nmid g$.

Thus these forms of restrictions on rational functions can create the needs for regular functions as we are now restricting to point P or a set of finite points.

Definition 1.3.2: Regular

If $u \in \mathbb{C}(X)$ and point $P \in X$, then u is regular at point P if u has an rational function such that $q(P) \neq 0$.

Definition 1.3.3: Regular Function

u is regular at all points $X \setminus \{q = 0\} \cap X$, with finitely many points.

Consider $x : \{x^2 + y^2 = 1\}$ and $u = \frac{1+y}{x}$. Is u regular when $x = 0$? We in fact claim that at $P = (0, -1)$, $\lim_{(x,y)\to(0,1)} u = \frac{1-y}{x} = \infty$, thus not regular.

Definition 1.3.4: $\mathbb{C}[X]$

Given X, the ring of regular functions, $\mathbb{C}(X) = \mathbb{C}[x,y]/(f(x,y)) = \mathbb{C}[X]$.

Definition 1.3.5: Integral

No zero divisors.

Definition 1.3.6: Fraction Field

$$\mathrm{Frac}(\mathcal{R}) := \{\frac{a}{b} : a, b \in \mathcal{R}, \frac{a}{b} = \frac{c}{d} \iff ad = bc\}$$

Lemma 1.3.3

For any integral ring, $\mathcal{R}$, there exists a corresponding function field Frac($\mathcal{R}$).

$$
\begin{aligned}
\mathcal{R} &= \mathbb{Z} \\
\mathrm{Frac}(\mathcal{R}) &= \mathbb{Q} \\
\mathcal{R}_1 &= \mathbb{C}[X] \\
\mathrm{Frac}(\mathcal{R}_1) &= \mathbb{C}(X)
\end{aligned}
$$

Recall that $\mathbb{C}(X) = \mathbb{C}(x)[y]/(f(x,y))$. This is another way of saying that this functions of rational functions in variable x. While the variable y is the coefficient in $\mathbb{C}(x)$.

Theorem 1.3.1

X is rational if and only if $\mathbb{C}(X) \cong \mathbb{C}(t)$ for some t.

1.4 Rational Maps

Let X, Y be irreducible plane curves. Recall that $u = \frac{p}{q} \in \mathbb{C}(X)$. Which is also well defined every from finitely many points of X.

$$u : X \longrightarrow \mathbb{A}^1$$

Given $u, v \in \mathbb{C}(X)$, we get $(u,v) : X \longrightarrow \mathbb{A}^2$.

Definition 1.4.1: Rational

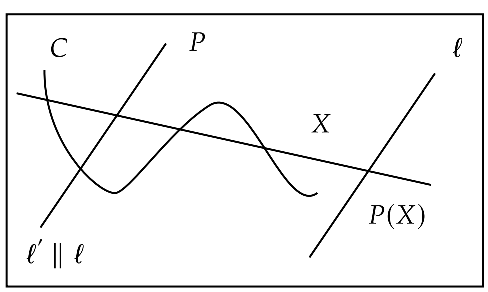
A map $\phi : XY$ is rational if and only if there exists rational functions $u, v \in \mathbb{C}(X)$ such that $(u,v) : X\mathbb{A}^2$ which factors through Y.

$$
\begin{tikzcd}
 & \mathbb{A}^2 \\
X \arrow[ur, dashed, "{(u,v)}"] \arrow[dr, dashed, "\varphi"'] & \\
 & Y \arrow[uu, "\iota"']
\end{tikzcd}
$$

Thus $(u,v) = \iota \circ \varphi$; i.e. $\mathrm{Im}(u,v) \subseteq Y$.

Any irreducible curve defines a rational map $\mathbb{A}^1 \to C$; i.e. if $C : \{f(x,y) = 0\} \subseteq \mathbb{A}$, then $x = \varphi(t)$ and $y = \varphi(t)$.

Let the line $\ell \subseteq \mathbb{A}^2$. Let $\ell \not\supseteq P$ be a point in $\mathbb{A}^2$.



Consider a projection map

$$
\begin{aligned}
P : \mathbb{A}^2 \setminus \ell' &\longrightarrow \ell \\
x &\longmapsto p(x)
\end{aligned}
$$

Consider $P|_C : C \longrightarrow \ell$, where C is any irreducible curve.

Definition 1.4.2: Birational

A rational map is birational if $\varphi : X \to Y$ if and only if there exists a rational map $\psi : Y \to X$ satisfying

(1) $\varphi \circ \psi = \text{Id}\, y$,

(2) $\psi \circ \varphi = \text{Id}\, x$,

whenever defined.

We call two curves birational if and only if there exists an birational map between them.

Theorem 1.4.1
X is rational if and only if $\mathbb{C}(X) \cong \mathbb{C}(t)$.

Proof: ($\Longleftarrow$). This is Easy.
($\Longrightarrow$). Suppose X is rational. Then we can parameterized $x = g(t)$ and $y = h(t)$ such that $\phi : \mathbb{C}(\mathrm{X}) \to \mathbb{C}(t)$ defined by $x, y \mapsto g(t), h(t)$. Check that ϕ is one-to-one. For surjectivity, we can use Luroth Theorem. ■

Theorem 1.4.2 Luroth Theorem
Let $K \subset \mathbb{C}(t)$ be a subfield, then there exists $r(t) \in \mathbb{C}(t)$ such that $K = \mathbb{C}(r(t))$ if we denote $r(t) = t^{'}$ implies that $k = \mathbb{C}(t^{'})$.

1.5 Singular and Non-singular Points

Let $C : \{f(x, y) = 0\} \subseteq \mathbb{A}^2$, $P \in C$ such that $f(P) = 0$.

Definition 1.5.1: Singular

P is singular if and only if $\partial f/\partial x(P) = 0$ and $\partial f/\partial y(P) = 0$.

Lemma 1.5.1
If $P = (0, 0) \in C$, then $f(x, y)$ has no singular term.

Lemma 1.5.2
If P is singular, then $f(x, y)$ also has no linear term.

Definition 1.5.2: Smooth

If for all $P \in C$, P is non-singular, then C is smooth.

Smooth is invariant under affine change in coordinates. It suffices to show that

$$x^2 - y^2 - 1 = 0 \quad \text{and} \quad x^2 - y = 0$$

are smooth.

Theorem 1.5.1
An irreducible curve C can only have finitely many singular points.

Being smooth depends on the defining function, which is talked about in Scheme Theory.

Proof of Theorem over $\mathbb{C}$: Let $P \in C$. Define $C^{'} : \{\partial f/\partial x = 0\}$. If P is singular, then $P \in C \cap C^{'}$. From the lemma, either $C \cap C^{'}$ is finite or C divides $C^{'}$; i.e. $f | \partial f/\partial x$. Since $\deg(\partial f/\partial x) < \deg(f)$ if and only if $\partial f/\partial y = 0$

which is the zero polynomial. True for $\partial f/\partial x$. But if $\partial f/\partial x = 0$ and $\partial y = 0$, then f is a constant. Remember since we are working over $\mathbb{C}$, we have the $\Gamma = 0$. Hence, this is not a curve C. ■

Definition 1.5.3: Multiplicity

In curve C, $P \in C$, assume $P(0,0)$, then $\text{Mult}(P)$ is the smallest degree monomial contained in f.

$$\begin{aligned} f(x,y) &= x - y + x^2 \\ &\implies \text{Mult}(0,0) = 1. \end{aligned}$$

Recall that $\text{Mult}(P) = 1$ if f contains a linear term, which implies that C is smooth at P. In fact, if C is smooth at P, $\text{Mult}(P) > 1$.

$$\begin{aligned} f(x,y) &= xy \\ &\text{Mult}(0,0) = 2. \end{aligned}$$

$$\begin{aligned} f(x,y) &= y^2 - x^3 \\ &\text{Mult}(0,0) = 2 \end{aligned}$$

If C is a curve of $\deg(n)$, then any point on C can have at most multiplicity n.

Chapter 2

Algebraic Varieties

2.1 Affine Varieties

Let $\mathbb{K} = \bar{\mathbb{K}}$ as in the closure such as $\mathbb{C}$ or $\bar{\mathbb{F}}_p$. We have already studied how $\mathbb{A}^n = \{(a_1, \ldots, a_n) : a_i \in \mathbb{K}\}$. Where $\mathbb{A}^1$ is a line, and $\mathbb{A}^2$ is a plane.

Definition 2.1.1: Zarisky Topology

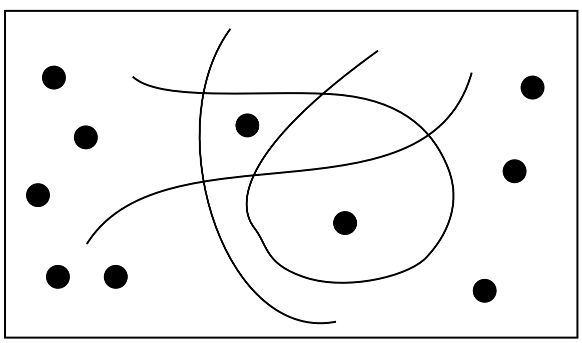
A closed subset of $\mathbb{A}^n$ is called a Zarisky Topology if $Z = \{f(x_1, \ldots, x_n) = f_2 = \ldots = 0\}$ labelled the zero set and $f_i \in \mathbb{K}[x_1, \ldots, x_n], i \in I$.

For example a Zarisky Topology in $\mathbb{A}^1$ consists of:

(1) $I = \phi$

(2) $f(x) = (x - \alpha_1)^{k_1} \ldots (x - \alpha_s)^{k_s}$, of finitely many zeros.

Where as a Zarisky Topology in $\mathbb{A}^2$:



This figure shows some points in the zero set that are of zero, one, and two dimensions. This can be very thin if closed and dense if open.

Proposition 2.1.1

Affine Varieties give a topology over $\mathbb{A}^1$

(1) ϕ when $f = 1 : \{1 = 0\}$.

(2) $\mathbb{A}^n$ when $I = \phi$.

(3) $\bigcap_{\alpha \in A} Z_\alpha$ such that $Z_\alpha = \{f_\alpha = 0\}$.

(4) $Z_1 \cup Z_2$ such that $\{f_{1j} f_{2j} = 0\}$.

***Proof of* (4):** Pick $P \in Z_1 \cup Z_2$, for the sake of contradiction suppose $Z \notin Z_1, Z_2$. Then there exists i such that $f_{1i}(P) \neq 0$ and there exists a j such that $f_{2j}(P) \neq 0$. Then $f_{1i} f_{2j}(P) \neq 0$. Hence a contradiction. ■

Lemma 2.1.1

A curve is non-singular if the dimension of the tangent space is the same dimension of the working space.

Definition 2.1.2: Ideal

An ideal $I \subset R$, is a commutative ring such that

1. a subgroup for addition : $i_1 + i_2 \in I$

2. $\iota \in R, i \in I \implies \iota i \in I$.

Then I is equal to the finite linear combinations of some function (f_i).

Proposition 2.1.2

We claim that if $\{f_i = 0\} = Z(I)$, then $I = (f_i)$.

How do we find uniqueness? Recall the following lemma:

Lemma 2.1.2

Any affine variety in $\mathbb{A}^n$, can be given in finitely many equations.

Proof: Suppose $Z = \{f_i = 0\}$, then $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian for all ideal I. $\{f_1 = 0\} \supset \{f_1 = f_2 = 0\} \supset \ldots$, then $(f_1) \subset (f_1, f_2) \subset \ldots$. Uniqueness follows from the fact that if we let $Z = Z(f_1, \ldots, f_n) = Z(I)$, which is a unique canonical ideal. ■

Definition 2.1.3: Noetherian

A topological space is Noetherian if and only if it satisfies the descending chain condition o closed subsets, i.e. for all $Y_1 \supseteq Y_2 \supset \ldots$ of closed subsets there is an integer r so that $Y_r = Y_{r+1} = \ldots$.

$\mathbb{A}^n$ is noetherian. If $Y_1 \supseteq Y_2 \supseteq \ldots$, then $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ is an acending chain of ideal in $A = \mathbb{K}[x_1, \ldots, x_n]$. Since A is noetherian ring it, this chain of ideal, is stationary. If for all i, $Y_i = Z(I(Y_i))$ so Y_i is stationary as well.

Proposition 2.1.3

In noetherian X, for all nonempty closed subset $Y = \bigcup_{i \in I} Y_i$ of irreducible closed subsets Y_i. If required $Y_i \not\supseteq Y_j$ for $i \neq j$, then Y_i is uniquely determined. They are irreducible components of Y.

Proof of Proposition: Let $\mathcal{E}$ be the set of nonempty closed subsets of X which cannot be written as a finite union of irreducible closed subsets. If $\mathcal{E}$ is nonempty, then since X i noetherian, it must contain a minimal element, say Y. Then Y is not irreducible, by contradiction of $\mathcal{E}$. Thus $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets of Y. By minimality of Y, each of Y' and Y'' can be expressed as a finite union of closed irreducible subsets, a contradiction arises once more. We conclude that all Y can be written as a union $Y = Y_1 \cup \ldots \cup Y_r$ of irreducible subsets. Assume $Y_i \supseteq Y_j$ for $i \neq j$.

Now suppose $Y = Y_1' \cup \ldots_s'$, then $Y_1' \subseteq Y = Y_1 \cup \ldots \cup Y_r$, so $Y_1' = \bigcup (Y_1' \cap Y_i')$. But Y_1' is irreducible, so $Y_1' \subseteq Y_i$ for some i, say $i = 1$. Similarly, $Y_1 \subseteq Y_j'$, for some j. Then $Y_1' \subseteq Y_j'$, so $j = 1$, thus $Y_1' = Y_1$. Now let $Z = (Y - Y_1)^-$. Then $Y_2 \cup \ldots \cup Y_r$, and $Z = Y_2' \cup \ldots \cup Y_s'$. By induction, we get uniqueness of Y_i. ■

Definition 2.1.4

Suppose R is commutative with 1, then $\sqrt{I} = \{f \in R : \exists f^k \in I\}$.

Lemma 2.1.3
$\sqrt{I}$ is an ideal.

Theorem 2.1.1 Hilbert's Nullstellensatz
Let $\mathbb{K}$ be an ideal, $a \in A = \mathbb{K}[x_1, \ldots, x_n]$ and $f \in A$ be a polynomial that vanishes at all $Z(a)$. Then $f^r \in a$ for some integer $r > 0$.

Mentions on the topic of Proof: This proof will consult many different theorems, which includes a paper by Daniel Allcock [**allcock**] ■

2.1.1 Hilbert's Nullstellensatz [allcock]

Theorem 2.1.2
Let $\mathbb{K}$ be a field and $\mathbb{F}$ be a field extension, finitely generated, as a $\mathbb{K} - algebra$. Then $\mathbb{F}$ is algebraic over $\mathbb{K}$.

Definition 2.1.5: Algebra

An algebra also called an algebra over a field, is a vector space equipped with a bilinear product.

Definition 2.1.6: Bilinear

Has distributivity and scalar multiplication.

Sketch of the Proof of the Theorem: Suppose $\mathbb{K}$ is infinite and $\mathbb{F}$ is the transcendental extension $\mathbb{K}(x)$. If $f_1, \ldots, f_m \in \mathbb{F}$, then the $\mathbb{K} - algebra$, A, is smaller than $\mathbb{F}$. To see this, choose a $c \in \mathbb{K}$ away from the poles of rational functions f_i. Then no element of A can have a pole at c, so $\frac{1}{x-c}$ is not in A and A is smaller than $\mathbb{F}$. ■

Definition 2.1.7: Pole

A pole of a function are values where the denominator is 0.

Proof of Theorem: Assume $\mathbb{F}$ is transcendental over $\mathbb{K}$, finitely generated as a $\mathbb{K} - algebra$, and we need to show that $\mathbb{F}$ is not finitely generated as a $\mathbb{K} - algebra$. A contradiction arose! Suppose $\mathbb{F}$ is transcendence degree one; meaning it contains a subfield $\mathbb{K}(x)$, a copy of a one-variable rational function field, and $\mathbb{F}$ is algebraic over $\mathbb{K}(x)$. With finite generation, then $\mathbb{F}$ has finite dimension. Choose a basis $e_1, \ldots, e_\ell$, such that

$$e_i e_j = \sum_k \frac{a_{ijk}(x)}{b_{ijk}(x)} e_k,$$

with $a, b \in \mathbb{K}[x]$. Let $f_0 = 1$ as a generator, such that

$$f_i = \sum_j \frac{c_{ij}(x)}{d_{ij}(x)} e_j,$$

with $c, d \in \mathbb{K}[x]$. So $a \in A$ is a $\mathbb{K} - linear$ combination of $f_0 = 1$ and products $f_1, \ldots, f_m$. Expanding the basis in a $\mathbb{K}(x) - linear$ combination of products of e_i, with case that denominators only include d's. a has coefficient involving b's and d's. Which means that if a is written in lowest terms, then b's and d's are being cancelled to the point where the denominator is not made of b's and d's. Thus:

$$\frac{1}{\text{a irreducible polynomial}} \notin A.$$

Thus A is smaller than $\mathbb{F}$. Thus infinitely many polynomials in $\mathbb{K}[x]$ will suffice for this to be true. If $\mathbb{F}$ has degree of trancendence greater than 1, thence change a subextension of $\mathbb{K}^t$ over which $\mathbb{F}$ has degree 1. By preserving $\mathbb{F}$ is not finitely generated as a $\mathbb{K}-algebra$, A is not a $\mathbb{K}-algebra$ either. ■

Definition 2.1.8: Trancendence Degree

The trancendence degree of extension field K over field $\mathbb{F}$, also known as t-deg(K), is the smallest number of elements that are not algebraic over $\mathbb{F}$.

Thus $\mathbb{Q}(\pi)$ and $\mathbb{Q}(\pi, \pi^2)$ is still of degree one. This is due to $\mathbb{Q}(\pi, \pi^2)$ makes π^2 algebraic over $\mathbb{Q}(\pi)$.

Theorem 2.1.3 'Weak' Nullstellensatz

Let $\mathbb{K}$ be an algebraically closed field. Then every maximal ideal in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ has the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in \mathbb{K}$. As a consequence, a family of polynomials on $\mathbb{K}^n$ with no common zero, generates the unit ideal of R.

Proof of 'Weak' Theorem: If m is a maximal ideal of R, then R/m is a field finitely generated as a $\mathbb{K}-algebra$. By the previous theorem, it is an algebraic extension of $\mathbb{K}$, hence equal to $\mathbb{K}$. Thus each x maps to some $a_i \in \mathbb{K}$ under $R \longrightarrow R/m = \mathbb{K}$, so m containing the ideal $(x_1 - a_1, \ldots, x_n - a_n)$. This is a maximal ideal, thus equal to m. To the second statement mentioned, consider the ideal generated by some given polynomial function with no common zeroes. If it is in a maximal ideal, m, then all functions would vanish at $(a_1, \ldots, a_n) \in \mathbb{K}^n$, contrary to hypothesis. Since it is not in m, it must be in all of R. ■

Theorem 2.1.4 Hilbert's Nullstellensatz

Suppose $\mathbb{K}$ is an algebraically closed field and g and $f_1, \ldots, f_m$ are in $R = \mathbb{K}[x_1, \ldots, x_n]$, as polynomial functions on $\mathbb{K}^n$. If g vanishes on the common zero-locus of f_i, then the same power of g lies in the ideal they generate.

Definition 2.1.9: Locus

The set of all points, whose location satisfies by condition.

Proof of Nullstellensatz: The polynomials $f_1, \ldots, f_m$ and $x_{n+1}g - 1$ have no common zeroes in $\mathbb{K}^{n+1}$. By Weak Nullstellsatz, we write:

$$1 = p_1 f_1 + \ldots + p_m f_m + p_{m+1}(x_{n+1}g - 1),$$

where the p's are polynomials in $x_1, \ldots, x_{n+1}$. Taking the image of this equation under the homomorphism $\mathbb{K}[x_1, \ldots, x_{n+1} \longrightarrow \mathbb{K}(x_1, \ldots, x_n)$ given by $x_{n+1} \mapsto 1/g$, we find:

$$1 = p_1(x_1, \ldots, x_n, 1/g)f_1 + \ldots + p_m(x_1, \ldots, x_n, 1/g)f_m.$$

Recall that $\mathbb{K}(x_1, \ldots, x_n)$ is a copy of an n-variable rational function field. After multiplying through by a power of g to clear the denominators, we have Hilbert's Theorem! ■

Corollary 2.1.1 Corollary of Nullstellensatz

There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals. (Ideals that are equal to their own radical in A). Given by $Y \mapsto I(Y)$ and $a \mapsto Z(a)$, a ideal. An algebraic set is irreducible if and only if it's ideal is prime.

Definition 2.1.10: Prime Ideal

A subset of a ring that shares the same properties of prime integers.

Definition 2.1.11: Irreducible Algebraic Set

An algebraic set that cannot be written as the union of two algebraic subsets, thus maximal.

Proof of Corollary: ($\Longrightarrow$). If Y is irreducible, $I(Y)$ is prime. If $fg \in I(Y)$, then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Thus $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both closed subsets of Y. SInce Y is irreducible, Y is either $Y = Y \cap Z(f)$ or $Y = Y \cap Z(g)$, thus $Y \subseteq Z(f)$ or $Y \subseteq Z(g)$. Thus $f \in I(Y)$ or $g \in I(Y)$.

($\Longleftarrow$). Let p be a prime ideal, suppose $Z(p) = Y_1 \cup Y_2$. Then $p = I(Y_1) \cap I(Y_2)$, so either $p = I(Y_1)$ or $p = I(Y_2)$. Thus $Z(p) = Y_1$ or $Z(p) = Y_2$, thus irreducible, thus maximal. ■

$\mathbb{A}^n$ is irreducible since it's corresponding to the zero ideal in A, thus prime.

Let f be a irreducible polynomial in $A = \mathbb{K}[x, y]$. Then f generates a prime ideal in A, since A is a unique factorization domain (UFD). So a $zero$ set $Y = Z(f)$ is irreducible. The affine curve is defined by $f(x, y) = 0$. If f has degree d, then Y is a curve of degree d.

Definition 2.1.12: Integral Domain

Non-trivial commutative ring where the non-zero products is non-zero for all x, y

Definition 2.1.13: Unique Factorization Domain

An integral domain where every non-unit can be written as a product is irreducible elements, unique up to order and units.

Lemma 2.1.4

A is a polynomial ring, thus a UFD.

Definition 2.1.14

Hence $I \subset R$ is radical if $\sqrt{I} = I$.

A counterexample for $\mathbb{K} = \mathbb{R}$ is in $\mathbb{A}^1 \supset Z = \{1 = 0\} = Z((1)) = \{x^2 + 1 = 0\} = Z((x^2 + 1))$. Thus $(x^2 + 1) \subset \mathbb{K}[x]$ is radical. For $f \in \mathbb{R}[x]$, if $f^k \in (x^2 + 1)$, then $f \in (x^2 + 1)$, $f^k(x^2 + 1) \cdot p(x)$. If $f = (x^2 + 1)g(x)$, since $\mathbb{R}[x]$ is a UFD and $x^2 + 1$ is irreducible.

Corollary 2.1.2

We can think of a corollary of the Nullstellensatz as

$$\{\text{Radical Ideals } J \subset \mathbb{K}[x_1, \ldots, x_n]\} \leftrightarrow \{\text{Zarisky Closed } S \subset \mathbb{A}^n_{\mathbb{K}}$$

2.2 Zarisky Topology

Lemma 2.2.1

An affine variety is a set of fractions such that it is a Zarisky closed subset adjoined with the ring of regular closed functions.

Definition 2.2.1: Regular Function

$Z \in \mathbb{A}^n$ is regular function if $f|_Z$ maps to $\mathbb{K}$.

$$I(Z) \overset{\text{kernel}}{\hookrightarrow} \mathbb{K}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{K}[Z] = \frac{\mathbb{K}[x_1, \ldots, x_n]}{I(Z)}.$$

Rings that appear this way are finitely generated algebras over $\mathbb{K}$, without nilpotence.

Definition 2.2.2: Nilpotence

$a \in R$ is nilpotent if $a \neq 0$ but $a^n = 0$.

If $R = S/I$, then $S \longrightarrow S/I$ and $f \mapsto \bar{f} = f \mod I$. $\bar{f} \neq 0 \iff f \notin I$, but $\bar{f}^n = 0 \iff f^n \notin I$.

Definition 2.2.3

$\mathbb{A}^n \supset X \longrightarrow \mathbb{A}^1 = \mathbb{K}$. A regular map $\pi : X \longrightarrow \mathbb{A}^1 \rightleftharpoons$ is a regular function on X.

$$X \longrightarrow \mathbb{A}^m = \mathbb{K}^m$$

Definition 2.2.4: Regular Map

$\mathbb{A}^n \subset X \xrightarrow{\pi} Y \subset \mathbb{A}^m$.

$$\pi : X \longrightarrow Y \text{ Is a regular Map}$$
$$\vec{\pi} : X \longrightarrow \mathbb{A}^m \text{ such that } \vec{\pi}(X) \subset Y$$

Definition 2.2.5: Isomorphism

A regular map that has a regular inverse.

Proposition 2.2.1

$$\begin{tikzcd} & & \mathbb{A}^m \\ \mathbb{A}^n \subset X \arrow[r] \arrow[urr] & Y \arrow[ur, hook] & \end{tikzcd}$$

Then $\pi : X \longrightarrow Y$ if and only if $\mathbb{K}[X] \longleftarrow \mathbb{K}[Y]$ is a homomorphism of $\mathbb{K}$-algebras.

Corollary 2.2.1

$$\{\text{affine varieties}\} \leftrightarrow \{\text{finitely generated } \mathbb{K}\text{-algebras without nilpotence}\}$$

Proof of Proposition: $\mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^m \supset Z(p) = \{p = 0\}$ where $p(y_1, \ldots, y_m)$ and $y_1 = f(x, \ldots, x_n)$. $\pi^{-1}(Z(p)) = \{p_1(f(x), \ldots, f_m(x)) = 0\} = Z(\pi^{-1}(p))$. This maps $\mathbb{K}[y, \ldots, y_m] \longrightarrow \mathbb{K}[x, \ldots, x_n]$ defined by $p \mapsto \pi^*(p)$. Hence $\vec{\pi}(x) \subset Y \iff X \subset \vec{\pi}^{-1}(Y) \iff \forall p \in I(Y), X \subset \vec{\pi}^{-1}(Z(p))$. Note that X is a mapping from $I(X) \longrightarrow \vec{\pi}^*(p)$. All of this is if and only if $I(X) \supset \vec{\pi}^*(I(Y))$.
For an abstract affine variety X the choice of $X \hookrightarrow \mathbb{A}^n$ if and only if a choice of algebraic generators $\mathbb{K}[X] \twoheadleftarrow \mathbb{K}[x, \ldots, x_n]$. Thus:

$$\frac{\mathbb{K}[x_1, x_2]}{x_2 - x_1^2} \cong \mathbb{K}[y]\mathbb{A}^n$$

■

Lemma 2.2.2 More about Zarisky Topology

(1) Closed subsets of affine varieties, X, are one to one to radical ideals on $\mathbb{K}[X]$.

(2) Basic closed and open sets.

(3) Regular Maps are continuous

(4) $\mathbb{K}[\mathrm{X}]$ has no zero-divisors

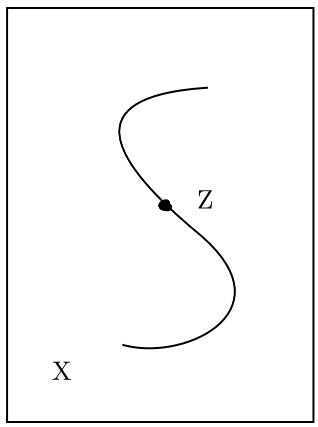
(5) $\mathbb{K}[\mathrm{X}]$ has no idempotence.

(6) Zarisky Topology is Noetherian

(7) $\mathbb{K}[\mathrm{X}] \twoheadleftarrow \mathbb{K}[Y] \iff X \xrightarrow{\pi} Y$ is a closed subset of Y.

(8) $\mathbb{K}[\mathrm{X}] \leftrightarrow \mathbb{K}[Y] \iff X \xrightarrow{\pi} Y$ if $\pi\bar{(}x) = Y$, meaning π is dominant.

Elaboration on (1): Suppose we have the topology, X, on $\mathbb{A}^n$ with point Z.



Note that $Z \subset X \subset \mathbb{A}^n$. Then we have $I(Z) \supset I(X)$.

Definition 2.2.6: Topology on set S

We have the closed sets Z_α and open sets U_α. Then the basis of a toplogy is $Z(f) = \{f = 0\}$ (closed) and $D(f) = \{f \neq 0\}$ (open).

Elaboration on (3):

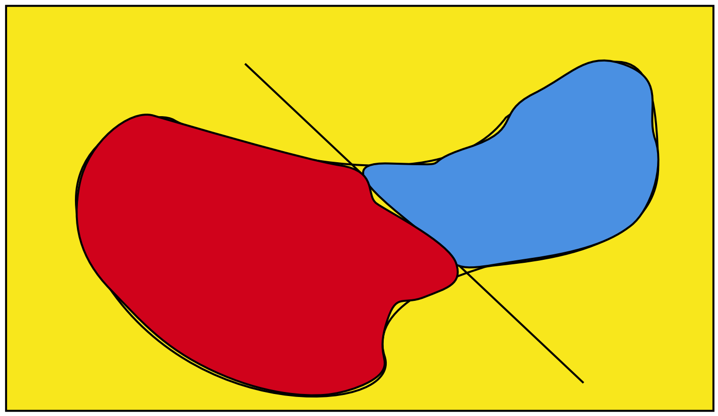
$$\begin{tikzcd} \mathbb{A}^n & \mathbb{A}^m \\ X \arrow[u] \arrow[ur] \arrow[r, "\pi"'] & Y \arrow[u] \\ & Z(f) \arrow[u, phantom, "\cup"] \end{tikzcd} \qquad \begin{array}{l} \mathbb{K}[\mathrm{X}] \leftarrow \mathbb{K}[Y] \\ \pi^*(f) \mapsfrom \quad f \end{array}$$

Recall that $\pi^{-1}(Z(f)) = Z(\pi^*(f))$

Definition 2.2.7: Irreducible

X is irreducible if and only if $X \neq Z_1 \cup Z_2$ such that $Z_i \subsetneq X$ is closed.

Elaboration on (4):
Recall that a normal classical topology in $\mathbb{R}$ is made up of the union of two subsets.



Note that the splitting of the topology is added to show the union of two subsets In fact, it is enough to consider **(4)** by showing when z_1, z_2 is basic. If $z_1 \cup z_2 = X$, then $Z(f_1) \cup Z(f_2) = X$.

Definition 2.2.8: Connected

The disjoint union of points that are open and closed.

Definition 2.2.9: Idempotence

If $e \in \mathbb{R}$ then $e^n = e$.

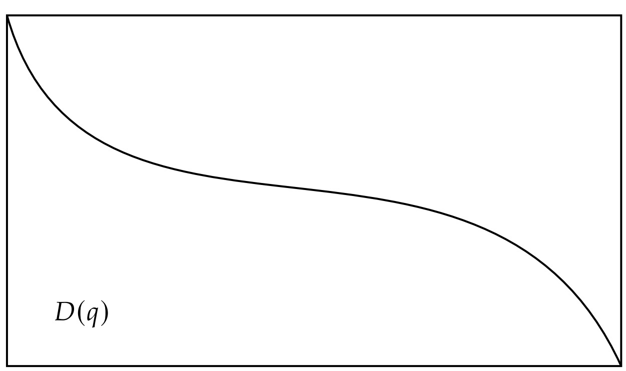
Lemma 2.2.3

For all $X = Z \cup \ldots \cup Z_k$, finitely many irreducible closed subsets, then $Z_i \subset Z_j$. Then we can define Z_i as the irreducible compositions of X.

Definition 2.2.10: Closure

Intersection of all closed sets.

Now assume that X is an irreducible affine variety if and only if $R = \mathbb{K}[X]$ has no zero-divisors. We already have the quotient field $\mathbb{K}[X] = \mathrm{Frac}(R)$, where $\varphi \in \mathbb{K}(X)$ is a function $\varphi : X \supset U \to \mathbb{K}$. Then we have



$Z(q) = \{q = 0\}$

Lemma 2.2.4

If X is irreducible, nonempty, open subsets $U_1, U_2 \subset X$, then $U_1 \cap U_2 \neq \phi$

Proof: $Z_1 \cup Z_2 = X$, then since X is irreducible, then Z_1 or Z_2 equals X. ■

2.3 Projective Varieties

Consider $U = \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2$. It is a fact that there is no natural way to give U the status of an affine variety. This is still affine if we remove a whole curve.

Definition 2.3.1: Localization

$$\begin{aligned}\mathbb{K}[Y] &= \frac{\mathbb{K}[x, \ldots, x_n]}{(I(X), y - f(x))} \\ &= \mathbb{K}[X][\frac{1}{f}]\end{aligned}$$

If $f \in \mathbb{R}, R[\frac{1}{f}]$.

Definition 2.3.2: $\mathbb{P}^n$

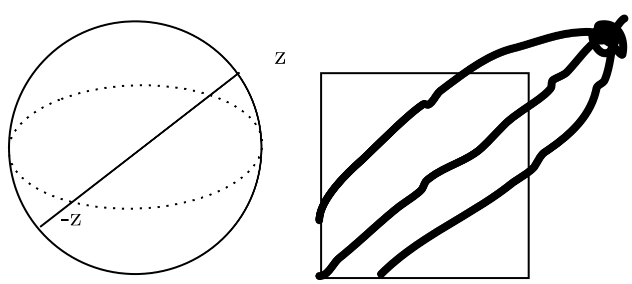
$$\frac{\{x_0, \ldots, x_n\} \in \mathbb{K}^{n+1} \setminus (0, \ldots, 0)}{\sim}$$

Designated as the points of $P^n \leftrightarrow$ lines in $\mathbb{A}^{n+1} \setminus (0, \ldots, 0)$.

Lemma 2.3.1

$$\mathbb{P}^n = (\mathbb{A}^n \cup \ldots \mathbb{A}^n) \setminus \{x_n \neq 0\}$$

$$\begin{aligned}\mathbb{P}^1 &= \mathbb{A}^1 \cup \{\infty\} \\ \mathbb{P}^n &= \mathbb{A}^n \setminus \{x_0 \neq 0\} \coprod \mathbb{A}^{n-1} \setminus \{x_1 \neq 0\} \coprod \ldots \coprod a^0 \setminus \{x_n \neq 0\} \\ \mathbb{P}^2_{\mathbb{R}} &= \mathbb{R}^2 \coprod \{\text{Direction of lines}\} = S/(z \sim -z)\end{aligned}$$



Note that for $\mathbb{P}^2_{\mathbb{C}}$, we have $\mathbb{C} - \dim(\mathbb{P}^2_{\mathbb{C}}) = 2$, but we have the $\mathbb{R} - \dim(\mathbb{P}^2_{\mathbb{C}}) = 4$. In fact:

Lemma 2.3.2

If X is an algebraic variety, and $\mathbb{C} - \dim = d$ but $\mathbb{R} - \dim = 2d$, then X is orientable.

Think of the following coordinates in $\mathbb{A}$ and in $\mathbb{P}$. x_0 makes sense in $\mathbb{A}^{n+1}$, but not in $\mathbb{P}^n$. However, $x_0 = 0$ makes sense in $\mathbb{P}^n$.

Definition 2.3.3: Zeroes in Projective Space

Note that $Z_+(p)$ is the well defined subset of $\mathbb{P}^n$, indicated projective by the plus subscript.

Definition 2.3.4: Homogeneous

If all monomials have the same degree d.

Definition 2.3.5: Graded Rings

$$\oplus_{d=0}^{\infty} R_d = R = \mathbb{K}[x_0, \dots, x_n]$$

such that $R_a \cdot R_b \subset R_{a+b}$.

Definition 2.3.6: Dehomogenization

$p \longrightarrow q$ defined by $(x_0, \dots, x_n) \mapsto (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$.

Lemma 2.3.3

$$\deg(q) = \deg(p).$$

2.4 Zarisky Topologies on Projective Varieties

Take a Zarisky closed subset of $\mathbb{P} * n \supset X$ be one to one with a radical ideal of $I \subset \mathbb{K}[x_1, \dots, x_{n+1}]$, where the polynomials are homogenous. But then we need to define the homogenous ideal.

Definition 2.4.1: Homogenous Ideal

Take a graded ring
$$R = \oplus_{d=0}^{\infty} R_d \supset I$$
I is homogenous if for all $f \in I$, then f can be decomposed into a sum of other $f' \in I$. Therefore $f = f_0 + f_1 + \dots + f_d$, up to degree d. This is true if and only if $Y \subset \mathbb{A}^{n+1}$ is canonical. Hence, I is generated by homogenous generators.

Definition 2.4.2: Canonical

If a line is made of all multiples of a single variable over the field.

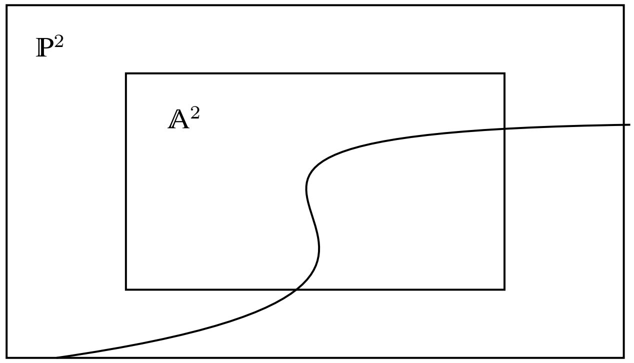
Lemma 2.4.1

If you take homogenous polynomials, $\mathbb{P}^n \ni p(x_1, \dots, x_{n+1})$, there is a bijective morphism that maps it to unhomogenous polynomials $(y_1, \dots, y_n \in \mathbb{A}^n$. Then $p \mapsto q(y_1, \dots, y_n) = p(1, y_1, \dots, y_n)$. The inverse is $x_0^d q(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$.

If we take $y\epsilon = y_1^2 + y_2^2 - 4$, then the homogenous inverse will be $x_0^2 \left(\left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 - 4 \right)$. Then $x_2/x_1 = \pm\sqrt{-1}$ or $\mathbb{P}^2 \setminus \mathbb{A}^2 = \{x_0 = 0\} = \{\infty\}$.

Lemma 2.4.2

A Zarisky topology on $\mathbb{P}^n|_{\mathbb{A}^n}$ is a Zarisky topology on $\mathbb{A}^n$.



Look at the following diagram: Now the question is: Is this open or closed? In fact, the image of a projective variety under a morphism of algebraic varieties is closed for Zariski topology (that is, it is an algebraic set). If $Y \in \mathbb{A}^n$, then $\bar{Y} \in \mathbb{P}^n$. Thus $I_+(\bar{Y}) = \{\hom(p) : \forall p \in I(Y)\}$.

We have previously defined affine variety zarisky closed subsets as a closed set being a part of $\mathbb{A}^n$. Note that it is a collection of closed balls around a point. Now take quasi-affine varieties as taking U in the affine space that is an closed neighbourhood, and removing atleast one zero. Thus it contains open and closed neighbourhoods. Similarly we have projective and quasi-projective varieties.

Lemma 2.4.3

If we take a rational function in quasi-projective varieties, $U \overset{open}{\subset} X \overset{closed}{\subset} \mathbb{P}^n$, then U is a regular function such that $f : U \longrightarrow R$ such that it is a neighbourhood of any point $V \in U$ such that f is made of a fraction of homogenous polynomials and same degree, where $q(V) \neq 0$.

Lemma 2.4.4 Sheaf Property

If $Y \in \mathbb{A}^n$ is an affine varieity, then the set of regular functions is the same as defined before.
Then the set of regular functions on $D(f) \subset Y$ is equal to $\mathbb{K}[Y][\frac{1}{f}]$.

Definition 2.4.3: Sheaf

A collection of local regular functions

Maps between quasi-projective varieties are a regular map $f : U_1 \longrightarrow U_2$ such that f is a map of sets such that on a neighbourhood of each point $v \in U_1$, polynomials of same degree $f(x_1, \ldots, x_{n+1} = (p_0(x_1, \ldots, x_{n+1}), \ldots, p_m(x_1, \ldots, x_{n+1}))$ and there exists $p_i(v) \neq 0$.

Corollary 2.4.1

Given $Y \supset D(f) = \{f \neq 0\}$, then $D(f) \cong$ to an affine variety.

Lemma 2.4.5

Suppose $U = X \setminus Z \subset \mathbb{P}^n$ is irreducible, $U \neq V_1 \cup V_2$, where $V_i \overset{closed}{\subsetneq} U$ for all two open $W_1, W_2 \subset U$ intersect and not empty. Then the rational functions on U for a field $\mathbb{K}(U)$.

Lemma 2.4.6

$$\{\text{affine varieties over } \mathbb{K}\}/\text{isomorphism} \leftrightarrow \{\text{finitely generated } \mathbb{K}\text{-algebras w/o nilpotence}$$

Which is a injective function up to direction and inverses. This is the same as

$$\{\text{quasi-projective irreducible varieties}\}/\text{birational isomorphisms} \leftrightarrow \{\text{finitely generated fields } K/\mathbb{K}\}$$

Definition 2.4.4: Rational Map

It is a regular map that is defined on an open subset.

A famous example is $\mathbb{P}^1 \to \mathbb{P}^3$ defined by $(x_0, x_1) \mapsto (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$.

Chapter 3

Algebraic Maps

3.1 Regular Maps

Let X be a quasi-projective variety with coordinates $[x_0, \ldots, x_n]$.

Definition 3.1.1: Regular Function

A regular function on X is a degree 0 rational homogenous function

Assume $\mathbb{A}^1 \subseteq \mathbb{P}^1$ where $\mathbb{A}^1 = \{x_0 \neq 0\}$. We already know that $\mathbb{A}^1 = \mathbb{C}[t]$. We claim that $\mathbb{C}[t]$ is isomorphic to a degree 0 rational function. Regular functions are regular maps.

Lemma 3.1.1

A regular maps between quasi-projective varieties X, Y is a map f such that for every point $x \in X$, there exists an open neighbourhood U of X in the Zarisky Topology, and an affine chart $\mathbb{A}^m \subseteq \mathbb{P}^m$ containing $f(x)$ with $f(U) \subseteq \mathbb{A}^m$ and $f|_U : U \to \mathbb{A}^m$ is regular.

Example 3.1.1 (The Veranese Embedding)

Fix $n, d \in \mathbb{N}$. The degree d Veranese embedding of $\mathbb{P}^n$ given by $f : \mathbb{P}^n \to \mathbb{P}^m$. We claim that $\binom{m+1=d+1}{d}$

Example 3.1.2 (The Cremona Map)

A map $\mathbb{P}^2 \to \mathbb{P}^2$ is defined by $[x_0 : x_1 : x_2] \mapsto [x_1x_2, x_0, x_2, x_0, x_1]$ is a rational map. It is regular on $\mathbb{P}^2 \setminus \{[1:0:0], [0:1:0], [0:0:1]\}$

Lemma 3.1.2

If X, Y are birational, then $C(X) \cong C(Y)$. Converse is true.

3.2 Finite Maps

Proposition 3.2.1

Let X, Y be affine varieties, then $f : X \to Y$ be regular such that $f(X)$ is dense in Y. Then $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is injective.

Proof: Given f and f^*, then assume $f^*(g_1) = f^*(g_2)$ for $g_1, g_2 \in \mathbb{C}[Y]$. Then $g_1 \circ f(X) = g_2 \circ f(X)$, thus g_1, g_2 agree on a dense subset of Y. Hence $g_1 = g_2$ on Y. ■

Definition 3.2.1: Finite Map

We call f regular such that $f(X) \subset Y$ is dense a finite map if $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$

Definition 3.2.2: Integral over a field

We say B is integral over A, A, B rings such that $A \subseteq B$, iff for all $b \in B$, there exists an equation

$$b^k + a_2 b^{k-1} + \ldots + a_k = 0$$

for some $k \in \mathbb{N}$, for all $a \in A$.

Let $X := \{y^2 = x\} \subseteq \mathbb{A}^2$, then $f : X \to \mathbb{A}^1$ is the projection map onto the $x - coordinate$. We claim that $\mathbb{C}[x, y]/(y^2 - x) = \mathbb{C}[X]$ is integral over $\mathbb{C}[Y] \setminus \mathbb{C}[X]$. We want to show that each element of $b \in \mathbb{C}[X]$ is a root of a polynomial integral, or check the generators of X and Y of $\mathbb{C}[Y]$.

Proof: If $b = x$, then $x - x = 0$. If $b = y$, then $y^2 - x = 0$. ■

Definition 3.2.3: Properties of finite maps

1. $f : X \to Y$, finite, then any $y \in f(X)$ has finitely many preimages. If Y is irreducible, the degree is the general number of preimages.
2. If f is finite, then f is surjective.
3. If f is finite and $C \subset X$ is closed, then $f(C)$ is closed.

3.3 Jouanolou's Trick

Theorem 3.3.1 Jouanolou's Trick
There exists an affine variety X with a regular map $X \hookrightarrow \mathbb{P}^n$ such that the fibers are an affine variety.

Proof with Dialogue:

Definition 3.3.1: Fiber

The preimage of a map: $f^{-1}(p)$.

Suppose $\mathbb{P}^1$ is the lines through $0 \in \mathbb{A}^2$. Consider all linear maps $\mathbb{A}^2 \to \mathbb{A}^2$ with image $l := \{y = 0\}$. Let M be a rank 1 matrix for the linear transformation. For example:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Definition 3.3.2: Rank

The number of linearly independent variales.

Moreover $M^2 = M$. Thus $\mathbb{A}^2 \cong \ker(M) \oplus \mathrm{Im}(M)$.

Hence we have two eigen values associated with $x^2 - x = 0$, 0 and 1. 0 is the eigenspace for $\ker(M)$ and 1 is the eigenspace associated with $\mathrm{Im}(M)$. Take $u \in \mathrm{Im}(M)$, then $u = Mv$ implies $Mu = Mv$. Thus $u = v$.

Define $X :=$ as a 2x2 matrix of M such that $M^2 = M$ and $\mathbf{rank}(M) = 1$. We claim that X can be naturally be viewed as an affine variety inside $\mathbb{A}^4$.

If $M^2 = M$, then we have the following four polynomials.

$$a^2 + bc = a \tag{3.1}$$
$$ab + bd = b \tag{3.2}$$
$$ac + dc = c \tag{3.3}$$
$$bc + d^2 = d \tag{3.4}$$

Any $2x2$ matrix can have rank $0, 1$ or 2. If $\mathbf{rank}(M) = 2$, then M is invertible. Thus we must have $\det(M) = 0$. To ensure $\mathbf{rank}(M) \neq 0$, we must remove $[0]$. We claim that X is disconnected, so $[0]$ is an isolated point. Consider the regular map

$$X \longrightarrow \mathbb{P}^1$$
$$M \longmapsto \mathrm{Im}(M)$$

Where $\mathrm{Im}(M)$ is a line in $\mathbb{A}^2 \subseteq \mathbb{P}^1$. Then $f : M \longrightarrow \det(Id_2 - M)$ is a characteristic polynomial that forms $f([0]) = 1$ and $f([M]) = 0$. Thus $\Gamma(poly) = 0$ meaning that $\det(I - M) = 0$ or $(1 - a)(1 - d) - bc = 0$. ■

3.4 Local Ring of a Variety at a Point

Let X be a quasi-projective variety, $x \in X$.

What are the properties of X near x?

Theorem 3.4.1

In a quasi-projective variety, every point has a neighbourhood isomorphic to an affine variety.

Note that affine sub-varieties are zarisky closed in $\mathbb{A}^n$, but when they are embedded in a projective space, they are Zarisky open.

Definition 3.4.1: Local Ring of X at x

Assume X is affine.
Our goal is to define $\mathcal{O}_{X,x}$ to be the local ring of X at x.
The construction of $\mathcal{O}_{X,x}$ is a special case of a general construction called "localization of a ring at a prime ideal" in commutative algebra.
Recall from Commutative Algebra. Let A be a commutative ring with 1, $p \subset A$ is a prime ideal. $A_p :=$ for $f, g \in A$ such that $g \notin p$, let $(f, g)/\sim$ be such that $(f, g) \sim (f', g')$ if there exists an $h \in A/p$ such that $h(fg' - f'g) = 0$.
Now to get $\mathcal{O}_{X,x}$, take $A = \mathbb{C}[X]$. Then $p = m_x$ is the maximal ideal given by regular functions on X vanishing at x. Hence

$$\mathcal{O}_{X,x} := A_p = \mathbb{C}[X]_{m_x}$$

We have a remark that there exists $\phi : A \longrightarrow A_p$ defined by $a \mapsto (a, 1)$ such that $a \notin p$ shows that $(a, 1) \in A_p^x$. So the common notation is $(a, b) = \frac{a}{b}$.

We also have a secondary remark that there exists a unique maximal ideal $m = \{\frac{f}{g} \in A_p : f \in p\}$.

Definition 3.4.2: Local ring

A ring with a unique maximal ideal is a local ring.

Definition 3.4.3: Alternative definition to $\mathcal{O}_{X,x}$

"The stalk of the structure sheaf" is where $\mathcal{O}_X$ is the structure sheaf, and $\mathcal{O}_X(U)$ is the ring of the regular function of U, e.g. $\mathcal{O}_X(X) = \mathbb{C}[X]$.
Hence, $\mathcal{O}_{X,x} := \lim_{U \ni x \to \infty} \mathcal{O}_X(U) \cong A_p$.

There is a warning of caution since if $p = m_x$, then it is the maximal ideal in A. But if $m = m_x$, then it is the maximal ideal in A_p.

Let $X = \mathbb{A}^1.\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t] = A$. Then x is a point corresponding to $0 \in \mathbb{A}^1$. A_0 is the local ring at $x = \{\frac{f}{g} : g(0) \neq 0\}$. For example, $\frac{1}{1-t} \in A_0$, but $\frac{1}{t} \notin A_0$; $m_x = \{\frac{f}{g} \in A_0 : f(0) = 0\}$ such that $\frac{1}{1-t} \notin m_0$ and $\frac{t^3}{1-t} \in A_0$ and in m_0.

Chapter 4

Geometric Points

4.1 Tangent Space

Let X be an affine variety with $x \in X$. Our goal is to define the tangent space of x. We will have two definitions referenced in this section, the concrete and intrinsic.

4.1.1 Concrete Definition

Assume $\mathrm{X} \subseteq \mathbb{A}^n$ and without loss of generality, let $x = \{\vec{0}\} \subseteq \mathbb{A}^n$. Our idea is that the tangent space at x, which is denoted by T_x, will be defined as the union of all lines that are tangent to X at x. A tangent space will exist at singular points, not just smooth points. Diversion of a tangent space won't necessarily agree. If this is the case, then what does it mean for a line to be tangent at x? Hence, calculus will not suffice. We will define the intersection multiplicity and use that to determine tangency.

Definition 4.1.1

A line is tangent at a point x if it has intersection multiplicity $\geqslant 2$ at x.

Proposition 4.1.1

Let ℓ be a line in $\mathbb{A}^n$ which passes through $x = (0, \ldots, 0)$. Pick a point $a = (a_1, \ldots, a_n) \subseteq \ell$. Then all other points on ℓ are of the form ta, for $t \in \mathbb{C}$. Let $\mathrm{X} : \{f_1 = \ldots = f_m = 0\} \subseteq \mathbb{A}^n$, note that $\deg(f_i) = n$. For each $1 \subseteq i \subseteq m$, consider f_i/ℓ is a polynomial in one variable, t. Furthermore, note that $f_i(0) = 0$, since $x = (0, \ldots, 0) \subseteq \mathrm{X}$. The same is true when restricted to ℓ. Then $f_i|_\ell = t^{k_i} g_i(t)$ such that $g(0) \neq 0$.

Definition 4.1.2: Multiplicity of x as a Root

Then $k_i \in \mathbb{Z}$ and we call k_i the multiplicity of x as a root of $f_i|_\ell$ or the order of vanishing of $f_i|_\ell$ at 0. If $f_i|_\ell = 0$, then we say $k_i = \infty$.

Definition 4.1.3: Intersection Multiplicity

We define the intersection multiplicity of ℓ with X at x as

$$\mu := \min_{1 \leqslant i \leqslant m} \{k_i\} \in \mathbb{Z}$$

Definition 4.1.4: Line of Tangency at x

This is true if the intersection multiplicity, $\mu \geqslant 2$.

Definition 4.1.5: Tangent Space

The tangent space at $x \in X$ is
$$T_x = \bigcup \ell,$$
given ℓ is tangent at $x \in X$.

$$X = \{y = x^2\} \subset \mathbb{A}^2$$

Pick an arbitrary ℓ and $a = (a_1, a_2) \subseteq \ell$. If ℓ is tangent to X at $(0,0)$:

$$f(x,y) = x^2 - y = 0$$
$$f|_\ell = (a_1 t)^2 - (a_2 t) = 0$$
$$f|_\ell = a_1^2 t^2 - a_2 t$$
$$\mu \geqslant 2 \iff a_2 = 0$$

Assume $a_2 = 0$. Then if $a_1 \neq 0$, then $\mu = 2$. Else if $a_1 = 0$, then we now have a point, not a line so we can ignore this case. Let $\ell = \{y = 0\} \subseteq \mathbb{A}^2$ is the tangent line in $\mathbb{A}^2$.

$$X := \{y(y - x^2) = 0\} \subseteq \mathbb{A}^2$$

What is T_x at $x = (0,0)$? Tempted to think that $y = 0$ is the tangent space, but it wont work. Let ℓ be a line through $(0,0)$, and fix a point (a_1, a_2) so ℓ is parameterized by $(a_1 t, a_2 t)$.

$$\begin{aligned} f|_\ell &= a_2 t(a_2 t - (a_1 t)^2) \\ &= a_2^2 t^2 - a_1 a_2 t^3 \end{aligned}$$

We claim that $\mu \geqslant 2$ for all a_1, a_2. If $a_1 = a_2 = 0$, then $\mu = \infty$. If $a_1, a_2 \neq 0$, then $\mu = 2$. If $a_1 = 0$ or $a_2 = 0$, then $\mu = 2$. Then any line through $(0,0)$ is a tangent line to X. Then $T_{(0,0)} = \mathbb{A}^2$.

$$\{y = ax\} \subseteq \mathbb{A}^2$$

$$\begin{aligned} f &= -ax \\ f(a_1 t, a_2 t) &= a_2 t - a a_2 t \end{aligned}$$

If $a_2, a_1 \neq 0$, then $\mu = 1$, but if a_1 or $a_2 = 0$, then $\mu = 1$ as well. If $a_1 = a_2 = 0$, then we can omit this case as no line, only a point. If $a_2 = a a_1$, then we have a the zero polynomial $\mu = \infty$. This is the case where $\ell := \{y = ax\}$. The tangent line is $y = ax$.

4.1.2 Intrinsic Definition

Let X be an affine variety with $x \in X$. We defined T_x as the union of all lines tangent to X at x. When we want to provide an intrinsic definition it really depends only on X and not on the embedding to $\mathbb{A}^n$. Recall that a local ring is $\mathcal{O}_x$ at x is only defined in terms of the ring of regular functions on x, which shows intrinsic. Let $\mathcal{O}_x \supseteq m_x$ be defined as the max ideal such that the elements are regular functions in $\mathcal{O}_x$, which vanishes at x. Consider $mx^2 = \{\sum a_i b_i : a_i, b_i \in mx\}$.

Definition 4.1.6: Cotangent Space

Given a vector space, the cotangent is all dot products of the space such that it is equal to 0. Therefore, all lines are perpendicular to the tangent space—denoted mx/mx^2.

Theorem 4.1.1

$$T_x = (mx/mx^2)^{\checkmark}$$

equivalently $T_x = mx/mx^2$.

We will show that the cotangent space of R is the vector space over the residue space R/m. Whose dual is the tangent space?

Definition 4.1.7: Lift

Given two morphism $f : X \to Y$ and $g : Z \to Y$, we say f lifts through Z by giving a morphism $h : X \to Z$ such that $f = (g \circ h)(x)$.

Sketch of Proof: Let $f \in \mathbb{C}[\mathbb{A}^n]$. Without loss of generality, assume $x = (0, \ldots, 0) \subseteq \mathbb{A}^n$. The differential of f at x is $dxf = \sum_1^n \frac{\partial f}{\partial t_i}(0)dt_i$: which is $(\mathbb{A}^n)^{\checkmark} \to$ linear functions of t_i mapped by $\mathbb{A}^n \to \mathbb{C}$.

So $dx : \mathbb{C}[\mathbb{A}^n] \to (\mathbb{A}^n)^{\checkmark}$: $f \mapsto dxf$. For $X \subseteq \mathbb{A}^n$ affine, $\mathbb{C}[X] = \mathbb{C}[\mathbb{A}^n]/I_x$. $X = \{f_1 = \ldots = f_m = 0\}$ which induces a map $\mathbb{C}[X] \xrightarrow{dx} (\mathbb{A}^n)^{\checkmark}/(dxf_1, \ldots, dxf_m)$.

Note that we can pick a lift of g to $\tilde{g} \in \mathbb{C}[\mathbb{A}^n]$ such that g will be defined by considering the quotient. Then $dx\tilde{g} \in (\mathbb{A}^n)^{\checkmark}$.

Different lifts $\tilde{g}_i, \tilde{g}_j$ differ by linear combinations $f_1, \ldots, f_m$, then $dx\tilde{g}_i, dx\tilde{g}_j$ differ by linear combinations $dxf_1, \ldots, dxf_m$. This follows since linear combinations $a_1f_1 + \ldots + a_mf_m$ such that $a_i \in \mathbb{C}[\mathbb{A}^n]$ differentiate to $dx(a_1f_1 + \ldots + a_mf_m) = (dxa_i)f_1(x) + a_1(x)(dxf_1) + \ldots = 0$.

We claim that $T_x^{\checkmark} \cong (\mathbb{A}^n)^{\checkmark}/(dxf_1, \ldots, dxf_m)$ if and only if $T_x = \{a \in \mathbb{A}^n : dxf_i(a) = 0, \forall i\}$. Hence we can view $dx : \mathbb{C}[X] \to T_x^{\checkmark}$ such that $T_x^{\checkmark} = mx/mx^2 = \tilde{mx}/\tilde{mx}^2$. Where $\tilde{mx}$ is the ideal in $\mathbb{C}[X]$ formed by regular functions vanishing at x. Note that $mx/mx^2 = \tilde{mx}/\tilde{mx}^2$.

Consider the restriction $dx|_{\tilde{mx}} : \tilde{mx} \to T_x^{\checkmark}$. To complete the proof we must show that $dx|_{\tilde{mx}}$ is subjective and has kernel $\tilde{mx}^2$. ■

Proof: We want any element in the dual of T_x is of the form dxf for $f \in \tilde{mx}$. We know $T_x = \mathbb{A}^n/(dxf_1, \ldots, dxf_m)$. Let $g \in T_x^{\checkmark} = (dxf_1, \ldots, dxf_m)$. Pick a lift $\tilde{g} = \mathbb{A}_n^{\checkmark}$ of g. We claim that $dx\tilde{g} = \tilde{g}$ since $\tilde{g}$ is linear.

$$
\begin{array}{ll}
\tilde{g} \in \mathbb{C}[\mathbb{A}^n] \xrightarrow[\tilde{g} \mapsto \tilde{g}]{dx} & (\mathbb{A}^n) \\
 & /(dxf_1, \ldots, dxf_m) \\
\tilde{g}|_x \in \mathbb{C}[X] \xrightarrow{dx} & (\mathbb{A}^n)/(dxf_1, \ldots, dxf_m) \\
\qquad\quad \cup| & \quad || \\
\tilde{g}|_x \in \quad \tilde{mx} \xrightarrow{dx|_{\tilde{mx}}} & (T_x) \to g
\end{array}
$$

$\ker(dx|_{\tilde{mx}} = \tilde{mx}^2$, such that $\tilde{mx}^2 \subseteq \ker(dx|_{\tilde{mx}}$. Hence if $\tilde{mx}^2 \ni h = \sum h_ih_j : h_ih_j \in mx$. Thus $dx|_{\tilde{mx}}(h) = \sum(dxh_i)h_j(x) + h_i(x)dx(h_j)$. ■

If $X = \mathbb{A}^1$, then $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[X]$. Then the tangent space defined by $\tilde{mx} = (x)$, but $\tilde{mx}^2 = (x^2)$ is isomorphic to $\tilde{mx}/\tilde{mx}^2 =\cong \mathbb{C}$ such that $\mathbb{C}a \leftarrow\!\shortmid a$.

4.2 Dimension and Singular/Nonsingular Points

Assume X is irreducible. Consider $\mathbb{C}(X)$.

Definition 4.2.1: Transcendence Degree

The dimension of X is the transcendence degree of the field extension $\mathbb{C}(X)$ over $\mathbb{C}$. Or this can also be defined as the maximum number of algebraically independent elements of $\mathbb{C}(X)$ over $\mathbb{C}$.

Definition 4.2.2: Algebraically Independent Elements

Take $\mathbb{C}(X) \ni f_1, \ldots, f_k$ are algebraically independent elements over $\mathbb{C}$. If there does not exist a polynomial $p(x_1, \ldots, x_k)$ such that $p(f_1, \ldots, f_k) = 0$.

For example, take $X = \mathbb{A}^1$ and $\mathbb{C}(X) = \mathbb{C}(t)$. We claim that t-deg($\mathbb{C}(t)$) over $\mathbb{C}$ is one.

Proof: t is algebraically independent over $\mathbb{C}$. But this only means that t-deg($\mathbb{C}$(X)) $\geqslant 1$. We want to show that t-deg($\mathbb{C}$(X)) $= 1$, by showing for all $f \in \mathbb{C}$(X), $\{f, t\}$ is not algebraically independent.
Say $f = \frac{p_1(t)}{p_2(t)} \in \mathbb{C}$(X). Let $p(x_1, x_2) = p_2(x_1)x_2 - p_1(x_1)$, then $p(t, f) = p_2(t)f - p_1(t)$. Then $\cancel{p_2(t)}\frac{p_1(t)}{\cancel{p_2(t)}} - p_1(t) = 0$.
Similarly t-deg($\mathbb{C}(\mathbb{A}^n)$) $= n$. ∎

Recall that X, Y are birational if and only if $\mathbb{C}(X) \cong \mathbb{C}(Y)$. Thus the dim $\mathbb{P}^n = n$ since $\mathbb{C}(\mathbb{P}^n) = \mathbb{C}(t_1, \ldots, t_n)$.

Theorem 4.2.1

Let $f(x_1, \ldots, x_n)$, nonzero, irreducible polynomials in x variables, then dim(X) $= n - 1$.

Lemma 4.2.1

The dimension of a hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ is $n - 1$.

Corollary 4.2.1

If X is not irreducible let X_i be the set of irreducible components of X such that dim(X) = max{dim(X_i)}.

Definition 4.2.3: Local Dimension

$$\dim_x(X) = \max(\dim(x_i)).$$

Theorem 4.2.2

For all $x \in X$, $\dim(T_x) \geqslant \dim_x(X)$.

Lemma 4.2.2 Nonsingular Points

$x \in X$ is nonsingular, or smooth if and only if $\dim(T_x) = \dim_x(X)$.

4.3 Blowups in Projective Space

First and foremost, let us preface this section by mentioning a couple of words on why we are looking into "blowups". We know that there exists a isomorphism between birational cuves and nonsingular projective curves... However, this is not particularly true for higher dimensional varieties.

Before we can address the information, let's do a little commutative review. Let $K \subseteq L$ be fields such that L is a field extension.

$$\deg_{\mathbb{K}}(L) = [L : \mathbb{K}] = \dim_{\mathbb{K}}(L)$$

Note that $\dim_{\mathbb{K}}(L)$ is the dimension of L in a $\mathbb{K}$ vector space. For example if we have $\mathbb{R} \subset \mathbb{C}$, then $\deg_{\mathbb{R}}(\mathbb{C}) = 2$. Then $\mathbb{C} \cong \mathbb{R}^2$, such that t-$\deg_{\mathbb{R}}(\mathbb{C}) = 0$. If $\deg_{\mathbb{K}}(L) < \infty$, then t-$\deg_{\mathbb{K}}(L) = 0$. In fact, every element of L is algebraic over $\mathbb{K}$. For all $x \in L$, there exists $p(t) \in \mathbb{K}[t]$ such that $p(n) = 0$. There exists N such that $1, x, x^2, x^2, \ldots, x^N$ linearly dependent, meaning $\sum a_i x^i$ for all i is 0. Hence a t-$\deg_{\mathbb{K}}(L) \neq 0$ implies $\deg_{\mathbb{K}}(L) = \infty$.

Definition 4.3.1: Krull Dimension

Let $X = \text{Spec}(\mathbb{C}[x])$ such that the t-deg($\mathbb{C}$(X)) has a one to one to the krull dimension of $\mathbb{C}[X]$.

Now the most simplest case of an birational curve that is not an isomorphism is a blowup.

4.3.1 Blowup at a Point

Shafrovich defines a blowup with points in $\mathbb{P}^n$ using homogenous coordinates, but it does get confusing. Therefore, we will rely on good old Hartshorne with affine coordinates. Our goal is to define the blowup of $\mathbb{C}^n$ at the origin, defined by $\mathrm{Bl}_O(\mathbb{C}^n)$.

Definition 4.3.2: Incidence Variety

Let $X : \mathrm{Bl}_O(\mathbb{C}^n)$ such that a subjective map of

$$X \hookrightarrow \mathbb{C}^n_{(x_1,\ldots,x_n)} \times \mathbb{P}^{n-1}_{[y_1:\ldots:y_n]}.$$

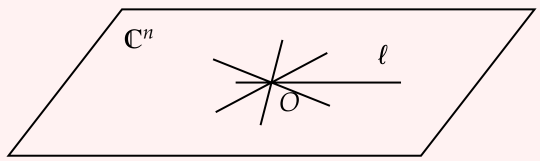


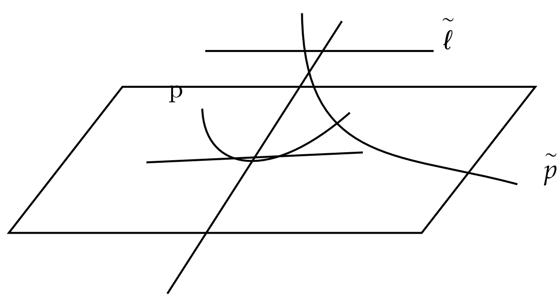
Figure 4.1: Set of lines ℓ through O in $\mathbb{C}^n$ such that $\{(pt, \ell)\} \in \mathbb{C}^n \times \mathbb{P}^{n-1}$

Then X := $\{(x, \ell) : \ell$ is a line through O, and x is a point of $\mathbb{C}^n$ which is on $\mathbb{P}$. This is called an incidence variety.

Then we have the cases such that if $x \neq O$, then there exists a unique line ℓ connecting O to x. Else we have every line through O passes through x. If we take $X \xrightarrow{\pi} \mathbb{C}^n$ defined by $(x, \ell) \mapsto x$. Then we take $\pi^{-1}(O) = \mathbb{P}^{n-1}$ and $\pi^{-1}(x) = pi(x, \ell) \in X$. But why is X defined by algebraic equations in $\mathbb{C}^n \times \mathbb{P}^{n-1}$. Better yet, if we had two points x and y, then how is x determined by y? Where y are the homogenous coordinates of ℓ up to scaling. Then we can pick a coordinate and view it in $\mathbb{C}^n$.
Let $x \in \ell$ if there exists t such that $x = ty$, i.e. $x_1 = ty_1$ and so forth. Then $x_i y_j = x_j y_i$ for all i, j. Hence the mapping $X \hookrightarrow \mathbb{C}^n \times \mathbb{P}^{n-1}$ is the locus of the equations. Generally, X is an irreducible variety of dimension n.

4.3.2 Blowup of a point on a quasi-projective variety

Assume X is an affine variety. $X \hookrightarrow \mathbb{C}^n$ defined by $x \mapsto O$. Then the blowup of $\mathbb{C}^n$ at O is $\mathrm{Bl}_O(\mathbb{C}^n) \xrightarrow{\pi} \mathbb{C}^n \hookleftarrow X$. What is $\pi^{-1}(X)$? $\pi^{-1}(X) = \pi^{-1}(O) \cup$



Note that if X is a smooth curve, then $\mathrm{Bl}_x(X) \cong X$, where x is a smooth point.

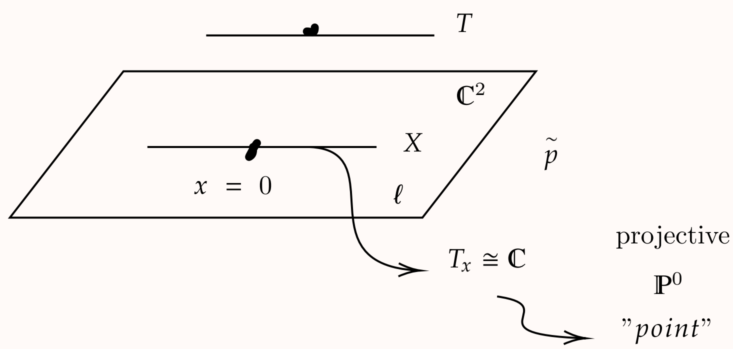
Lemma 4.3.1
$\mathrm{Bl}_x(X)$ is independent of the embedding $X \hookrightarrow \mathbb{C}^n$.

Lemma 4.3.2
If X is any quasi-projective variety, we can consider an affine chart around $x \in X$, then $\mathrm{Bl}_x(X)$ is still quasi-projective.

Lemma 4.3.3

Consider the blowup morphism $\mathrm{Bl}_x(X) \xrightarrow{\pi} X$.

- π is an isomorphism on $X \setminus \{x\}$.
- If $x \in X$ is a smooth point, then $\pi^{-1}(x)$ is isomorphic to the projectivization of the tangent space T_x.



Definition 4.3.3: Tangent Cone

The tangent cone of a variety at a point is a geometric object that approximates the variety near that point. Consider $x \in X$, x point, then a tangent cone at x is denoted $\mathcal{E}_x$.

Let I be the ideal of polynomials that vanish on $C(X)$. To find the tangent cone at x, translate x to the origin, then consider the polynomials in I. For each polynomial, take the terms of the lowest degree (leading terms) after translation. It is also associated with the graded ring by the leading terms of the elements in the ideal at x. These leading terms have an homogenous ideal, which also defines a cone in affine space. To think of it intuitively, we can think of the tangent cone as the limit of rescalings of $C(X)$ near x. If P is smooth, then the cone is just the normal tangent space. But if x is singular, which we will see the following lemma follows.

Lemma 4.3.4

If $x \in X$ is not a smooth point, then $\pi^{-1}(x)$ is the projectivization of the tangent cone, $\mathcal{E}_x$ of X at x.

$$\mathcal{E}_x \subseteq T_x = \mathbb{C}^n$$

Where $\mathcal{E}_x$ is an affine subvariety through $\mathcal{O}$, defined by homogenous polynomial equations $x \in X \hookrightarrow \mathbb{C}^m$ defined by $x \mapsto \mathcal{O}$. Then $X := \{f_1 = \ldots = f_k = \mathcal{O}\}$ which all vanish at $\mathcal{O} = x \in X$.
Then they decompose each f_i into homogenous polynomial. Let $f_i^{'}$ be the lowest degree non-zero homogenous polynomial contained in f_i. Then $\mathcal{E}_x = \{f_1^{'} = \ldots = f_k^{'} = \mathcal{O}\} \subseteq T_x = \mathbb{C}^n \subseteq \mathbb{C}^m$.
Since the equations of T_x are given by degree one parts of all defining equations $f_1, \ldots, f_k$ of X. Thus $\dim \mathcal{E}_x = \dim_x X$.

For example, if we have $y^2 = x^3$, a right opening cusp, then the degree is one. Hence $T_x = \mathbb{C}^2$ and the $\mathrm{Proj}(\mathcal{E}_x) =$ a point. Hence if we take $\mathcal{E}_x = \{y^2 = 0\}$, then the blowup is a smooth curve isomorphic to $\mathbb{A}^1$. Hence $\pi^{-1}(\mathcal{O}) =$ a point.

4.4 Normal Varieties

4.4.1 Integrally Normal

Let $A = \mathbb{C}[x,y]/(y^2 - x^2 - x^3)$. We claim that A is not integrally closed. Recall that if $K = p/q$, then $t = y/x \in K$ is integrally closed as it is not divisible by $y^2 - x^2 - x^3$. Thus $t^2 = \frac{y^2}{x^2} = \frac{x^2+x^3}{x^2} = 1 + x \in A$. Thus $t^2 - 1 - x = 0$, which is a constant in A. Hence polynomials in t with coefficients in A and leading term equalling 1, means that t is integral. But $t \notin A$.

If $p/q \in \mathbb{C}(X)$, then assume (p/q) is integral.

$$\begin{aligned}(p/q)^n + a_1 p^{n-1} q^{n-1} + \ldots + a_n q^n = 0 \\ \implies p^n = -(a_1 p^{n-1} q^{n-1} + \ldots + a_n q^n) \\ \implies q/p^n \implies q/p.\end{aligned}$$

Thus p/q can only be $a_1 \in \mathbb{C}[X]$ if $n > 1$, where $\mathbb{C}[X]$ is integrally closed.

Definition 4.4.1: Normal

An irreducible affine variety, X, is normal if and only if $\mathbb{C}[X]$ is integrally closed.

Generally an irreducible quasi-projective variety X is normal if and only if every point has a normal affine neighbourhood.

Lemma 4.4.1

If X is an irreducible quasi-projective variety, then X is normal if and only if $\mathcal{O}_x$ is integrally closed for all $x \in X$.

Lemma 4.4.2

X is normal if and only if every finite birational is an isomorphism.

Theorem 4.4.1

An irreducible curve is normal if and only if it is smooth.

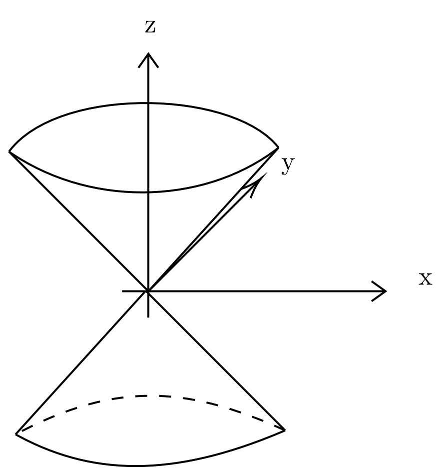
Theorem 4.4.2

If X irreducible quasi-projective is smooth, then it is normal.

Theorem 4.4.3

If X, irreducible quasi-projective, is normal then the set of singular points in X has height $\geqslant 2$

$$\{x^2 + y^2 = z^2\} =: S \subseteq \mathbb{C}^3$$



We claim that S is normal.

Proof of Example: With $u, v \in \mathbb{C}(x,y)$, $\mathbb{C}(X)$ has elements of the form $u + vz$. $\mathbb{C}[X]$ has elements of the form $u + vz = \mathbb{C}(X)$. Hence $\mathbb{C}[X]$ is finite over $\mathbb{C}[x.y]$, hence all integral over $\mathbb{C}[x,y]$. If $\alpha = u + vz \in \mathbb{C}(X)$ is integral over $\mathbb{C}[X]$, then it is also integral over $\mathbb{C}[x,y]$. Its minimal polynomial is

$$T^2 - 2uT + u^2 - (x^2 + y^2)v^2;$$

hence $2u \in \mathbb{C}[x,y]$, so $u \in \mathbb{C}[x,y]$. Similarly $u^2 - (x^2 + y^2)v \in \mathbb{C}[x,y]$, but $x^2 + y^2 = (x + iy)(x - iy)$. Hence it is the product of two irreducible coprimes, then $v \in \mathbb{C}[x,y]$. Therefore $\alpha \in \mathbb{C}[X]$. ∎

4.4.2 Normalization

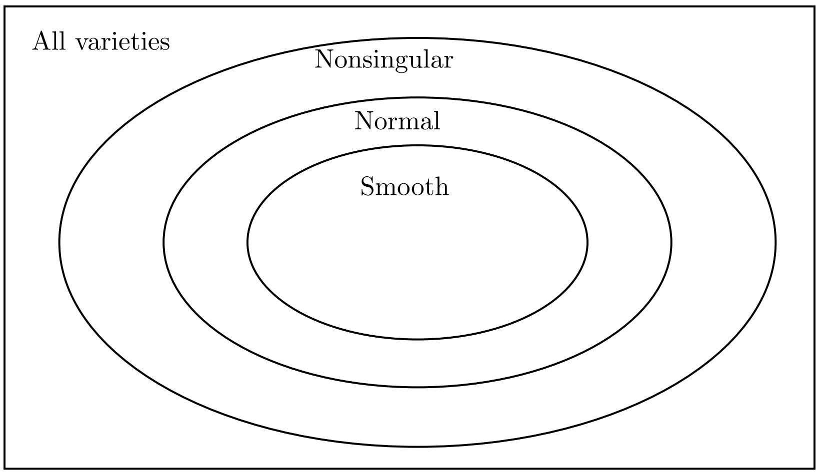


Figure 4.2: From Mumford's Red Book of Varieties and Schemes

Proposition 4.4.1

Let $Y = \mathrm{Spec}(S)$ be normal. $R \subset S$ is a finitely generated subring. S/R is a finite dimension. $X := \mathrm{Spec}(R)$ is not normal.

Let $S = \mathbb{C}[x,y]$, then $\mathrm{Spec}\, S = \mathbb{A}^2$. R is an ideal of 2 points, $(0,0), (1,0)$. Then $0 \to R \to S \to S/R \to 0$. Then $S/R \cong \mathbb{C} \oplus \mathbb{C}$.

Definition 4.4.2: Normalization

X is an irreducible quasi-projective variety. A normalization of X is a normal irreducible variety denoted by X^ν together with the normal map.

$$\pi : X^\nu \to X$$

is finite and birational. If X is not irreducible, then take $X = \bigcup X_i$ of irreducible components. $X^\nu = \bigsqcup X_i^\nu$.

Theorem 4.4.4 Geometric Noether Normalization Theorem (NNT)

Let X be an affine variety of dimension n. Then there exists a finite morphism

$$\pi : X \hookrightarrow \mathbb{A}^n.$$

Definition 4.4.3: V(S)

For every subset $S \subset R$, we have

$$\begin{aligned} V(S) &= \{x \in \mathrm{Spec}(R) : f(x) = 0 \text{ for all } f \in S\} \\ &= \{|\mathfrak{p}| : \mathfrak{p} \text{ is a prime and } \mathfrak{p} \supseteq S\} \end{aligned}$$

Proof: We can reduce this proof to the case of $P = \sqrt{(f)}$, or geometrically, the case of $Z = V((f))$. Suppose we have the decomposition

$$\sqrt{(f)} = P \cap P_1' \cap \ldots \cap P_t'$$

in R. If $Z_i' = V(P_i')$, then $Z, Z_1', \ldots, Z_t'$ are the components of $V((f))$. Pick an affine open $U_0 \subset X$ such that

$$\begin{aligned} &U_0 \cap Z \neq \phi \\ &U_0 \cap Z_i' = \phi, \qquad i = 1, \ldots, t \end{aligned}$$

Let $U_0 = X_g$, where

$$g \in P_1' \cap \ldots \cap P_t', \qquad g \notin P.$$

Then replace X by U_0, R by $R_{(g)}$, and in the new setup

$$\begin{aligned} V_{U_0}((f)) &= V_X((f)) \cap U_0 \\ &= Z \cap U_0 \end{aligned}$$

is irreducible; hence in $R_{(g)}$, $\sqrt{(f)} = P \cdot R_{(g)}$ is prime. We can now use the normalization lemma to find a morphism

$$X \xrightarrow{\pi} \mathbb{A}^n$$

$$R \xhookleftarrow{\pi^*} \mathbb{C}[X] = S$$

Let K be the quotient field of R and K/L is a finite algebraic extension. Let

$$f_0 = N_{K/L}(f).$$

Then we claim $f_0 \in S$ and

$$P \cap S = \sqrt{(f_0)}.$$

If we prove, the theorem follows. For R/P is an integral extension of $S/S \bigcap P$, so

$$\text{t-deg}_{\mathbb{C}}\, R/P = \text{t-deg}_{\mathbb{C}}\, S/(S \cap P).$$

But S is a UFD, so the primary decomposition of a prime ideal is just the product of the decomposition of the generator into irreducible elements. Therefore $P \cap S$ implies that f_0 is a unit times f_{00}^{ℓ} for some integer ℓ and some irreducible f_{00}, and that $P \cap S = (f_{00})$. Hence

$$\text{t-deg}_{\mathbb{C}}\, S/(S \cap P) = \text{t-deg}_{\mathbb{C}}\, \mathbb{C}[X]/(f_{00}) - n^{-1}$$

We check that first $f_0 \in P \cap S$. Let

$$Y^n + a_1 Y^{n-1} + \ldots + a_n = 0$$

be the irreducible equation satisfied by f over the field L. then f_0 is a power $(a_n)^m$ of a_n. Moreover, all the a_i's are symmetric functions in the conjugates of f; therefore the a_i are elements of L integrally dependent on S. Therefore, $a_i \in S$. In particular $f_0 = a_n^m \in S$, and since:

$$\begin{aligned} 0 &= a_n^{m-1}(f^n + a_1 f^{n-1} + \ldots + a_n) \\ &= f(a_n^{m-1} f^{n-1} + a_n^{m-1} a_1 f^{n-2} + \ldots + a_n^{m-1} a_{n-1}) + f_0, \end{aligned}$$

and $f_0 \in P$ also. Finally suppose $g \in P \cap S$. then $g \in P$, hence $g^n = fh$ for some integer n and some $h \in R$. taking the norms, we find that

$$\begin{aligned} g^{n[K:L]} &= N_{K/L}(g^n) \\ &= N_{K/L}(f) N_{K/L}(h) \in (f_0) \end{aligned}$$

since $N_{K/L} h$ is an element of S, by the reasoning used before. Therefore $g \in \sqrt{(f_0)}$, and $P \cap S$ is proven. ∎

Theorem 4.4.5

The normalization of any quasi-projective varieties exists and is unique up to isomorphism compatible with π

Sketch of Proof: Assume X affine. Let $A := \{f \in \mathbb{C}(X) : \text{f is integral over } \mathbb{C}[X]\}$. A is a subring of $\mathbb{C}(X)$, thus there is no zero divisors in A, and it is finitely generated over $\mathbb{C}$, by Noether Normalization Theorem (NNT). Then A is the ring of regular functions on an affine variety. Let $X := \text{Spec}(A)$. We claim that X^ν is the normalization of X, such that $X^\nu \xrightarrow{\phi} X$ induced by $\mathbb{C}[X] \hookrightarrow A$. In order to prove this, we must check that φ is birational and finite using the Noether Normalization Theorem. ■

If we let $X := \text{Spec}(\mathbb{C}[x, y]/(y^2 - x^3)$, then $\mathbb{C}[x]$ is not normal.

4.5 Singularities

Suppose $f : X \to Y$ is a regular map, where X, Y are quasi-projective varieties. The questions that arises are if X is irreducible, then what are the fibers of f irreducible, if they are respectively smooth. Generally this is not true. For example

$$\mathbb{A} \xrightarrow{f} \mathbb{A}$$
$$x \mapsto x^2$$

Where f^{-1} is not irreducible.

Theorem 4.5.1 First Bertini's Theorem

Assume X, Y irreducible and $f(x)$ is dense in Y, also known as f is dominant.

Suppose X remains irreducible over algebraic closure $\mathbb{C}(\bar{Y})$, then there exists $U \subseteq Y$ open and dense, such that for all $y \in U$, we have $f^{-1}(y)$ irreducible.
If we have the previous example of $\mathbb{A} \to \mathbb{A}$, and we have $\mathbb{C}[Y] = \mathbb{C}[t]$, then $\mathbb{C}(Y) = \mathbb{C}(t)$, which is not algebraically closed. Since we do not have $t^{1/2}$ or even if we unionize this, then we will not have $t^{1/3}$ and so forth.

Theorem 4.5.2

$$\overline{\mathbb{C}(t)} = \bigcup_{n \geqslant 1} \mathbb{C}(t^{1/n})$$

only true in Γ.

How can we view X as an irreducible variable in $\overline{\mathbb{C}(X)}$? Assume X, Y affine, then f induces $\mathbb{C}[Y] \to \mathbb{C}[X]$ by $\phi \to \phi \circ f$. Thus $\mathbb{C}[X]$ is a $\mathbb{C}[Y]$ algebra.
Note that being an algebra is a vector space equipped with a bilinear product.

$$\mathbb{C}[Y] \subseteq \mathbb{C}(Y) \subseteq \overline{\mathbb{C}(y)}.$$

$$\mathbb{C}[X] \bigotimes_{\mathbb{C}[Y]} \overline{\mathbb{C}(Y)}$$

Definition 4.5.1: Tensor Product

An image of a small element in C results in $A \bigotimes_C B$ as $ca \otimes b = a \otimes cb$.

Thus if we let $\tilde{X} = \text{Spec}\, \mathbb{C}[X] \bigotimes_{\mathbb{C}[Y]} \overline{\mathbb{C}(Y)}$, where $\tilde{X}$ is affine. Then in the statement of First Bertini's Theorem, we have that $\tilde{X}$ is irreducible if and only if X is irreducible over $\overline{\mathbb{C}(Y)}$.

Consider $\tilde{X} = \operatorname{Spec} \mathbb{C}[X] \bigotimes_{\mathbb{C}[t]} \overline{\mathbb{C}(t)}$. We claim that $\tilde{X}$ is not irreducible. Since $\mathbb{C}[x] \cong \mathbb{C}[t][x]/(x^2 - t)$ and this is a $\mathbb{C}[t]$ algebra. Then $\mathbb{C}[t][x]/(x^2 - t) \bigotimes_{\mathbb{C}[t]} \overline{\mathbb{C}(t)}$. Then $\overline{\mathbb{C}(t)}[x]/(x^2 - t)$ follows from tensor product. We can now factor due to closure. Thus this is the same as $\overline{\mathbb{C}(t)}[x]/(x - \sqrt{t})(x + \sqrt{t})$. Then $\operatorname{Spec} \overline{\mathbb{C}(t)}[x]/(x - \sqrt{t})(x + \sqrt{t}) = \tilde{X}$ is only two points, hence not irreducible.

The first Bertini's theorem ensures irreducibility over points $U \subseteq Y$ open and dense, but not all of Y.

Theorem 4.5.3 Second Bertini's Theorem

Given f, X, Y are quasi-projective, $f(x)$ is dense in Y, and X is nonsingular, then there exists a $U \subseteq Y$ open and dense, such that for all $y \in U$, $f^{-1}(y)$ is smooth.

We can think of this theorem as being the algebro-geometric analog of Sard's Theorem from differential geometry.

Theorem 4.5.4 Sard's Theorem

Take M, N as manifolds such that the set of critical values of f has measure zero in X.

Definition 4.5.2: Critical Points

X is a critical point if and only if

$$d_x f : T_x M \to T_{f(x)} N$$

not surjective.

Chapter 5

Differential Forms

5.1 Divisors

Suppose X is quasi-projective irreducible variety.

Definition 5.1.1: Prime Divisor

A prime divisor in X is an irreducible subvariety of X of height 1.

If $X = \mathbb{A}^1$ or $\mathbb{P}^1$, a prime divisor is a single point. If X is a surface, then the prime divisor is a curve.

Definition 5.1.2: Weil Divisor

A Weil Divisor D in X is given by $D = k_1C_1 + \ldots + k_rC_r : k_i \in \mathbb{Z}$, and C_i prime divisors.

We define $\text{Div}(X) := \{D(C(X))$ which is an abelian group. We define ***support*** of D to be the union of all C_i.

Definition 5.1.3: Effective

D is effective if for all i $K_i \geqslant 0$, not all 0.

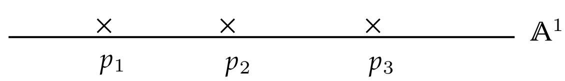


Figure 5.1: Defining points for the Divisor on $\{f_i = 0\} = X \subset \mathbb{A}^1$,
then we have our Divisor defined by $3p_1 - p_2 + 5p_3$.

Thus $D = \{\frac{f_1^3 f^5}{f_2}\}$ as the defining equation for D.

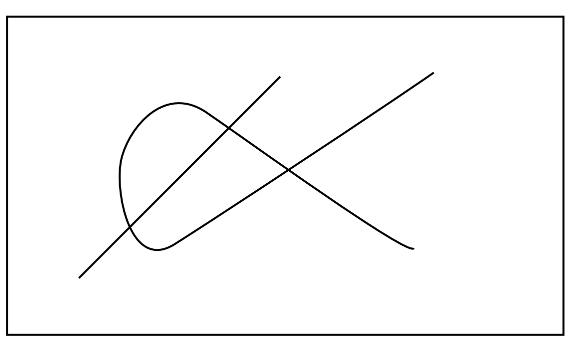
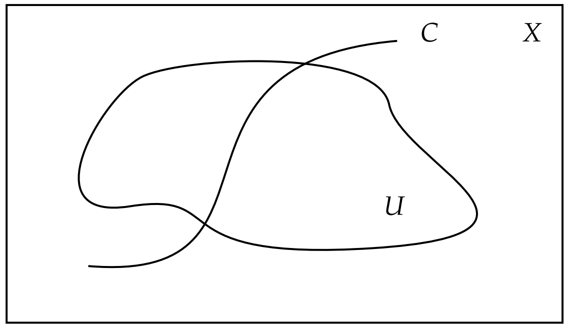


Figure 5.2: All points on the line and curve

We state our goal, given $f \in \mathbb{C}(X) \setminus \{0\}$, we define what $\operatorname{div}(f) \in \operatorname{Div}(X)$ is. Assume X is a nonsingular in height 1. Which means $\operatorname{Sing}(X)$ has height $\geqslant 2$, meaning X is normal. Given f, we define the order of vanishing $\nu_C(f)$ along a prime divisor $C \subset X$. Pick $U \subset X$, affine and dense, such that $C \subsetneq X \setminus U$. $C \cap U$ is a prime divisor defined by $\{\pi = 0\}$.



$\frac{g}{h} = f|_U \in \mathbb{C}(U)$ is a unique integer $k \in \mathbb{Z}^{\geqslant 0}$ such that $g \in (\pi)^k$, but $g \notin (\pi)^{k+1}$. Then k is the largest integer such that $\pi^k \mid g$. We set $\nu_C(g) = k$. Define similarly $\nu_C(h)$. Finally set $\nu_C(f) = \nu_C(g) - \nu_C(h)$. We remark that $\nu_C(f)$ is independent of the choice $U \subset X$.

If $X = \mathbb{A}^1$, then $\mathbb{C}(X) = \mathbb{C}(t)$. Suppose $f(t) = \frac{(t+3)^2(t-2)}{(t+1)^4}$. If nothing divides the numerator or denominator, then the order of vanishing is 0. In this case, we can split $f(t)$ into 3 subfunctions and find the order of vanishing on each subfunction, which is a curve as well. Where we can now get the definition of $\operatorname{div}(f)$.

Proposition 5.1.1

For any $f \in \mathbb{C}(X) \setminus \{0\}$, there exists only finitely many prime divisors.

$$\nu_f(C) \neq 0$$

Definition 5.1.4: Divisor of f

$$\operatorname{div}(f) = \sum \nu_C(f_i) C$$

Where C is a prime divisor.

Definition 5.1.5: Principle Divisor

$D \in \operatorname{Div}(X)$ is principle if and only if there exists f such that $\operatorname{div}(f) = 0$.

Considering the f from before, then we obtain $\operatorname{div}(f) = 2\{t = -3\} + \{t = 2\} - 4\{t = -1\}$. In fact, any point on $\mathbb{A}^1$ is a principle divisor but not in $\mathbb{P}^1$, as we can take the affine chart, which we know a principle divisor covers all of, but then we are now missing the points at infinity. Thus we only have a principle divisor in the projective space if we take the affine chart with the vanishing point at infinity. We find this intuition by thinking of the projective space as $\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_U$, where $\infty = \{U = 0\} \subsetneq \mathbb{A}^1_U$.

$$\begin{aligned} \nu_{\{\frac{1}{t}=0\}}(\infty) &= -1 \\ &= \operatorname{div}(f) = \{t = 0\} - \{t = \infty\} \end{aligned}$$

5.2 Divisor Class Group

Definition 5.2.1: Divisor Class

$$\begin{aligned} Cl(X) &= \text{Div(X)/Principle Divisor} \\ &= \text{Div(X)}/\sim . \end{aligned}$$

Where the relation denotes linear equivalence.

If D and $D' \in \text{Div}(X)$ with $D - P'$ principle, then we call D and D' linearly equivalent.

For example, $Cl(\mathbb{A}^n) = 0$, since divisors are principle. Given $D = \sum k_i C_i \subset \mathbb{A}^n$, where $C_i = \{H_i = 0\}$, H_i is a homogenous polynomial, then $f = \prod_i H_i^{k_i} = 0$ then $D = \text{div}(f)$.

However, $Cl(\mathbb{P}^n) = \mathbb{Z}$. Let $D = \sum k_i C_i$, then $f \in \mathbb{C}(\mathbb{P}^n)$ such that $\deg(f) = 0$. Then $\deg(D) = 0$, hence D is principle.

Proposition 5.2.1

A $\deg(D) \neq 0$, D is not principle, such that there is a natural map with X a quasi-projective:

$$\begin{aligned} \text{Div(X)} &\to \mathbb{Z} \\ D &\mapsto \deg(D) \\ \sum k_i H_i &\mapsto \sum k_i \deg(H_i) \end{aligned}$$

The class group of X, quasi-projective, is not finitely generated. X is a smooth projective curve of genus $g > 0$. When $X \neq \mathbb{P}^n$, there exists points $P \in X$ with $P \nsim Q \in Cl(X)$. If $P \sim Q$, then $P \nsim Q = \text{div}(f)$ for some $f \in \mathbb{C}(X)$.

$$f : X\mathbb{C} \subsetneq \mathbb{P}^1$$

Generally, this finite between two curves. Then $\text{div}(f) = P \sim Q$, then f has a simple zero of P, meaning multiplicity one, and a simple pole at Q. Then $\deg(f) = 1$ and f is birational and $X \cong \mathbb{P}^1$. For $P, Q \in X$ and $P \nsim Q$, then $Cl(X)$ is not finitely generated. Since ther exists uncountably many points on X.

5.3 Local Divisors

Definition 5.3.1: Cartier Divisor

This is a locally principle Weil Divisor.

Let $X = \bigcup U_i$, an affine open cover, $D \in \text{Div}(X)$ is called locally principle if $D|_{U_i}$ is principle.

If $X = \mathbb{P}^1, D = (0)$, is not principle, but it is locally principle if we can pick an affine chart $\mathbb{A}^1_t = \{t = 0\}$, then $\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_u$ such that $t = \frac{1}{u}$. $D = \{t = 0\}$ is principle in $\mathbb{A}^1$, then $D \cap \mathbb{A}^1_u = \phi$. D is locally principle.

A ***Group of Cartier Divisors*** is denoted $Ca(X)$. $CaCl(X) = Ca(X)/\text{principle divisors}$. People care about this group since $CaCl(X) \cong \textbf{Pic}(X)$ known as the Picard group of X, which is all line bundles up to isomorphism.

Theorem 5.3.1

If X is smooth, then every Weil divisor is Cartier.

An example of a Weil Divisor that is not Cartier, would be $X = \{xy = z^2\} \subseteq \mathbb{C}^3$, where $X \cong \{x^2 + y^2 = z^2\}$. This is equivalent to taking the hyperbolic conic:

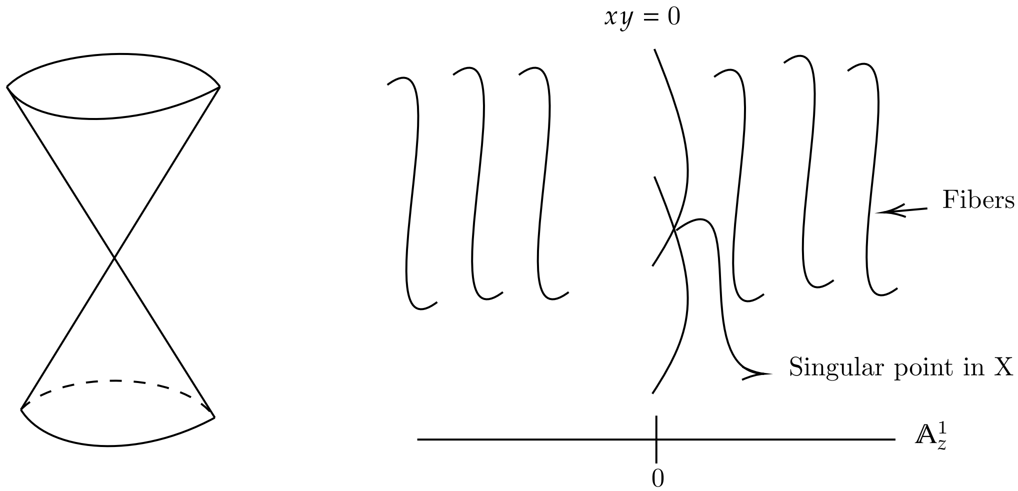
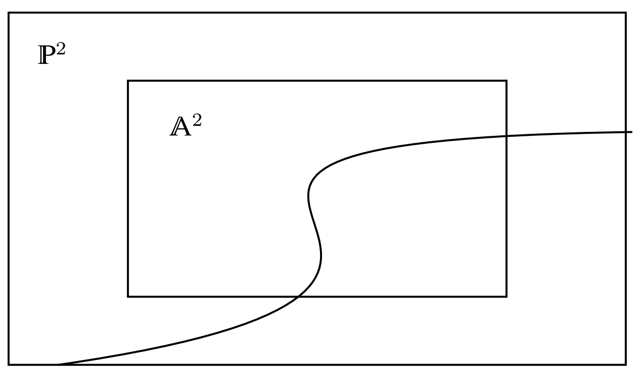


Figure 5.3: Taking an affine curve in $\mathbb{A}^3$, and breaking it into $\mathbb{A}^1_z$, where we see fibers on all points except the intersection of all three axes, which is a singular point.

We can see that $f^{-1}(0) = \{xy = 0\} = \{x = 0\} \cup \{y = 0\}$. Let $D = \{y = 0\} \subseteq f^{-1}(0)$. We claim that D is not Cartier. Then $\operatorname{div}(y = f) \neq D$, where $\nu_D(y) = D$. One can show $\nu_D(y) = D$. One can show $\nu_D(y) = 2$. Then $xy = z^2$, then $y = \frac{z^2}{x}$. Thus $D = \{y = 0\} = \{z = 0\}$, then $\nu_D(y) = 2$. We would rather want to look at the $\{z = 0\}$, as the ideal of the function vanishing at D is generated by z, not y. Hence $\operatorname{div}(f) = 2D$. When defining $\operatorname{div}(f) = \sum \nu_D(f)D$, we can take $U \subset X$, $C \cap U = \{\pi = 0\}$.



Take π such that the ideal of functions vanishing on D is generated by π.

5.3.1 Q-Factorial

Definition 5.3.2: Q-Factorial

X, a quasi-projective variety, is a Q-Factorial if for every Weil Divisor, there exists $n \in \mathbb{N}$ such that nD is Cartian.

Recall if we let $X = \{xy = z^2\}$, $\operatorname{div}(y) = 2D$, which is Cartier. If we let X be an affine ***toric variety***, an algebraic variety with a torus embedding, for example, $\mathbb{A}^n$. X is Q-Factorial if and only if it's ***fan***, a collection of rational polyhedral cones, is ***simplicial***, a set of points, lines, and triangles, and their n-dim counterparts.

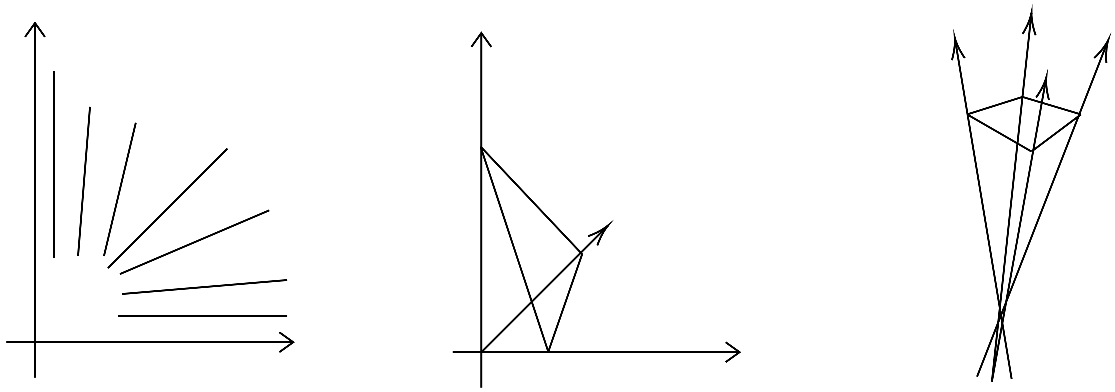


Figure 5.4: From left to right: $\mathbb{A}^2$, $\mathbb{A}^3$, $\mathbb{A}^4$: A conifold singular which is not Q-Factorial

5.4 Linear Systems of Divisors

Let X be a nonsingular variety. All Weil Divisors are Cartier in nonsingular varieties. If we have $D \subseteq X$ divisor, there is an associated vector space $\mathcal{L}(D)$ to be defined as

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) \setminus \{0\} : \operatorname{div}(f) + D \geqslant 0\}$$

$\operatorname{div}(f) + D = \sum a_i C_i$, then $a_i \geqslant 0$ for all i and there exists an i such that $a_i \neq 0$. Thus $\operatorname{div}(f) + D$ is effective. If $D = \sum n_i C_i$, then $f \in \mathcal{L}(D)$ if and only if $\nu_{C_i}(f) \geqslant -n_i$ and $\nu_f(C) \geqslant 0$ for any $C \neq C_i$. Thus if it is Cartier, then there exists $\mathcal{O}(D)$, a line bundle, $\mathcal{L}(D) = H^\circ(X, \mathcal{O}(D))$, where H° represents the space of sections of $\mathcal{O}(D)$. If $X = \mathbb{P}^1$ and $D = nX_\infty$, then say the point at infinity is at $[1:0]$. Then we ask, what is $\mathcal{L}(D)$?

Let $f \in \mathbb{C}(\mathbb{P}) \setminus \{0\}$, then $f = p/q$, where p, q are homogenous polynomials of the same degree, say d. Let $\{p, \ldots, p_d\}$ denote the set of zeros of $p(x_0, x_1)$, denoted $Z(p)$ and similarly $Z(q)$. So $\operatorname{div}(f) = \sum p_i - \sum q_i$. We want $\operatorname{div}(f) + D \geqslant 0$, where $D = nX_\infty$. If ther exists q such that $q_i \neq X_\infty$, then this is not true. Otherwise:

$$\begin{aligned} \operatorname{div}(f) + D &= \sum p_i - dX_\infty + nX_\infty \\ &= \sum p_i - (n-d)X_\infty \qquad \text{effective} \\ \iff & \ n - d \geqslant 0. \end{aligned}$$

Then $q = cX_i$, we have $f = \sum_0^d a_i X_0^i X_1^{d-i} / cX_1^d$. If $a_i' = a_i/c$, then $\sum_0^d a_i'(x_0/x_1)$. SO $f \in \mathbb{C}[x_0/x_1]_n$, meaning highest degree of n. Hence $\mathcal{L}(D)$ is a $\mathbb{C}$-vector space of $\dim n+1$. If $n \leqslant -1$, then $\mathcal{L}(D)$ is just a point $\{0\}$.

5.4.1 Hypersurfaces

Theorem 5.4.1

If X is a smooth, projective variety, then $\mathcal{L}(D)$ is of finite dimension.

Let $\dim_{\mathbb{C}}(\mathcal{L}(D)) = \ell(D)$, notationally. Why do we need projectivity. Let D be a 0-divisor, meaning if $D = \sum a_i C_i$, then all $a_i = 0$. Then $\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) \setminus \{0\}$ such that $\operatorname{div}(f) \geqslant 0\}$, such that $\operatorname{div}(f)$ is effective. So then f has no poles, thus f is regular. If $X = \mathbb{P}^1$, then $\mathcal{L}(D) = \mathbb{C}[\mathbb{P}^1] = \mathbb{C}$. If $X = \mathbb{A}^1$, then $\mathcal{L}(D) = \mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t]$. Then $\ell(D) = \infty$.

Definition 5.4.1: Linear System

Let X be projective, then the linear system of $D \in X$ is the projectivization of $\mathcal{L}(D)$, denoted $\mathbb{P}(\mathcal{L}(D))$. This is isomorphic to $\frac{\mathcal{L}(D)\setminus\{0\}}{\mathbb{C}^* = \mathbb{P}^{\ell(D)-1}}$.

Proposition 5.4.1

Suppose X is projective. If $D \sim D'$, then $\mathcal{L}(D) \cong \mathcal{L}(D')$.

In particular, $\mathbb{P}(\mathcal{L}(D)) = \mathbb{P}(\mathcal{L}(D'))$.

Proof: If $D \sim D'$ if and only if $D - D' = \operatorname{div}(g)$ for some $g \in \mathbb{C}(X)$. Consider $\phi : \mathcal{L}(D) \to \mathcal{L}(D')$ defined by $f = f \circ g$. Since $f \in \mathcal{L}(D) : \operatorname{div}(f) + D \geqslant 0$, $\operatorname{div}(f)$ effective. $\operatorname{div}(fg) + D' \geqslant 0$. To calculate it you find all intersecting vanishing order which are $\operatorname{div}(f) + \operatorname{div}(g)$.

$$\begin{aligned} \operatorname{div}(fg) + D' &= \operatorname{div}(f) + \operatorname{div}(g) + D' \geqslant 0 \\ &= \operatorname{div}(f) + (D - D') + D' \geqslant 0 \\ &= \operatorname{div}(f) + D \geqslant 0. \end{aligned}$$

Which holds true. Conversely $\phi^{-1} : \mathcal{L}(D') \to \mathcal{L}(D)$ is defined by $h \mapsto \frac{1}{g}h$ ■

One can check that $\mathbb{P}(\mathcal{L}(D)) \cong \{$ All effective divisors by linear equivalence$\}$. Let $f \in \mathcal{L}(D) \setminus \{0\}$, then $D = \operatorname{div}(f) + D \geqslant 0$. Moreover, cf for $c \in \mathbb{C}^*$ gives $\operatorname{div}(cf) = \operatorname{div}(f)$. Then D' remains the same. Let $X = \mathbb{P}^n$ and $D = \{f_d = 0\}$ be a hypersurface of degree d, and D is an effective divisor. Let D' be defined similarly. $D - D' = \operatorname{div}(\frac{f_d}{f'_d})$, since $\deg(f_d) = \deg(f'_d)$. Then $\mathcal{L}(D) \cong \{$ degree d homogenous polynomials in $x_0, \ldots, x_n\}$. $\frac{f_d}{f'_d} \in \mathbb{C}(\mathbb{P}^n)$. It is important that we already know how many elements are in the degree d hom.polys. set defined earlier, using stars and bars which we learned already. $\mathbb{P}(\mathcal{L}(D)) = \mathbb{P}^{\binom{n+d}{d}-1} = \{$ all hypersurfaces of degree d in $\mathbb{P}^n$.

Let D : cubic curve in $\mathbb{P}^2$. $\mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}^9$. We will show this divisor class group on D in $\mathbb{P}^n$ with genus g curve is not countable, with Riemann-Roch $Cl(\mathbb{P}^n) = \mathbb{Z}$ and $Cl(\sum p)$ is not countable. This is not subtle.

5.5 Degree of a Divisor

Let X be any smooth projective curve. Let $D = \sum a_i p_i \in \operatorname{Div}(X)$

Definition 5.5.1: Degree of a Divisor

Is the sum of a_i.

Let $X = \mathbb{P}^1$. A divisor D is principle if and only if $\deg(D) = 0$.

$$Cl(X) = \frac{\operatorname{Div}(X)}{p\operatorname{Div}(X)} \xrightarrow{\sim} \mathbb{Z}$$

Defined by $[D] \mapsto \deg(D)$

Theorem 5.5.1

Let X be a smooth projective curve. If D is principle, then $\deg(D) = 0$

Proof: D is principle implies $D = \operatorname{div}(f) = \sum v_{p_i}(f)p_i$ for some $f \in \mathbb{C}(X)$. So $D = \sum Z(f) - \sum P(f)$. We can rephrase the theorem so that $f \in \mathbb{C}(X)$ has $\deg(Z(f)) = \deg(P(f))$. Suppose f is non constant, then we have a rational map $X \xrightarrow{f} \mathbb{C} \setminus \{0\}$. This extends to a morphism $X \xrightarrow{f} \mathbb{P}^1$, which must be a finite map, or topologically a ramified cover. Then $\deg(f) = \#$ of preimages of a general point.

We claim $\deg(f) = \#$ of preimages of any point if we count multiplicity. For any $p \in \mathbb{P}^1$, $y \in f^{-1}(p) \subset U \subset X$ such that $f|_U : Z \to Z^k$, for some $k \in \mathbb{N}$. ■

The degree of a divisor is well defined up to linear equivalence. Therefore there exists an isomorphism $\deg : Cl(X) \to Z$. Thus $Cl^0(X) = \ker(\deg)$. There exists an exact sequence

$$0 \to Cl^0(X) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

Implying that $Cl(X) \cong \mathbb{Z} \oplus Cl^0(X)$.

Theorem 5.5.2
If $X \neq \mathbb{P}^1, Cl^0(X) \neq 0$, moreover, uncountable.

We claim that $Cl^0(X) \cong \mathscr{J}(X)$. For X being any genus g curve implies the jacobian of X being $g \dim \ell$ abelian variety which is a smooth projective variety with an algebraic group structure. As a complex manifold $\frac{\mathbb{C}^g}{Z^{2g}}$.

5.6 Bezout's Theorem

Definition 5.6.1: Hypersurface

A hypersurface $H \in \mathbb{P}^n$ is a projective variety defined, $H = \{F = 0\}$ for some $F \in \mathbb{C}[x_0, \ldots, x_n]$, homogenous.

Theorem 5.6.1 Bezout's Theorem
Let $X \subset \mathbb{P}^n$ be a smooth projective curve. $H \subset \mathbb{P}^n$ is not containing X, hypersurface, then $X \cdot H = (\deg(X))(\deg(H))$.

If $H = \{F = 0\}$, then $(\deg(H)) = \deg(F) = d$.

Definition 5.6.2: Hyperplane

If H' is a hyperplane, a projective linear subspace of height 1 in $\mathbb{P}^n$, that intersects with X transitivity, then $\deg(X) = \#X \cap H$.

How do we define $X \cdot H$? We can use divisors with $H = \{F = 0\}$, then $\mathrm{div}(F) \subseteq X$.

Definition 5.6.3

X and H' meet transversely at $P \in X \cap H'$ if

$$T_P(X) + T_P(H') = T_P(\mathbb{P}^n).$$

But this makes no sense. F is not a rational or regular function in $\mathbb{P}^n$ Look at the affine charts of $\mathbb{P}^n$, then we can dehomogenize F in said charts to work locally.

(1). Assume X is not contained in $\{X_i = 0\}$. If X is contained in $\{X_i = 0\}$, we can do a change in coordinates on $\mathbb{P}^n$.

(2). Set $U_i = \{x_i \neq 0\}$, note $\mathbb{P}^n = \bigcup_0^n U_i$ is a standard affine open cover. Let $f_i = \frac{F}{X_i^d}|_{X \cap U}$ f_i is regular on $X \cap U_i$.

(3). Consider $\mathrm{div}(f_i)$ on $X \cap U_i$ then this principle divisor i, effective since f_i is regular. Moreover, the division $\mathrm{div}(f_i)$ is supported on $X \cap U_i \cap H$.

$$\mathrm{Div}(X \cap U_i) \longrightarrow \mathrm{Div}(\overline{X \cap U_i \cap U_j})$$

This implies there exists a unique divisor $\mathrm{div}(F)$ as all of X such that restricting to $X \cap U_i$ gives $\mathrm{div}(f_i)$.

Definition 5.6.4: Intersection Number

The intersection number of X and H, denoted $X \cdot H$ is the degree of $\mathrm{div}(F)$ on X where $H = \{F = 0\}$.

This can be better stated as $X \cdot H = \deg(X) \deg(H)$.

Proof: Let H, H' be linearly equivalent in $\mathbb{P}^n$. Recall $H \sim H'$ if and only if $\deg H = \deg H'$.

We claim that $X \cdot H = X \cdot H'$. Assume $H := \{F = 0\}$ and similarly for H' then $\mathrm{div} F \sim \mathrm{div} F'$. Then $\mathrm{div} F - \mathrm{div} F' = \mathrm{div} \frac{F}{F'}$ where F, F' are homogenous polynomials in coordinates of $\mathbb{P}^n$, $\deg F = \deg F'$. So $\frac{F}{F'} \in \mathbb{C}(\mathbb{P}^n)$ and $\frac{F}{F'}|_X \in \mathbb{C}(X)$ implies $\mathrm{div}(\frac{F}{F'}|_X)$ is a principle divisor on k. Thus $\mathrm{div}(F) \sim \mathrm{div}(F')$.

On a smooth projective curve, linearly equivalent divisors have the same degree. Hence $X \cdot H = X \cdot H'$. So, to find $X \cdot H$, replace H with any linearly equivalent hyperspace H'. Take $H' = H_1 \cup H_2 \cup \ldots \cup H$, by factorizing F' into its linear polynomials. Let $H_i =$ the zero locus of such polynomial. So $X \cdot H' = X(H_1 \cup H_2 \cup \ldots \cup H) = \sum_1^{\deg(H)} X \cdot H_i = \deg(X)$. Thus $X \cdot H' = \deg(X)\deg(H)$. ■

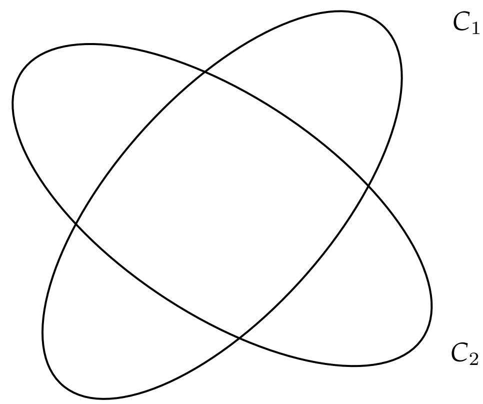


Figure 5.5: "General" means they intersect at 4 points.

Then their degree is 4. But how about arbitrary general curves?

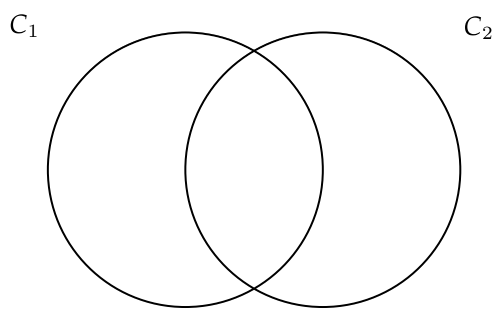


Figure 5.6: $C_1, C_2 \subseteq \mathbb{P}^2$

Then let $\mathbb{P}^2_{[x,y,z]}$ such that $z \neq 0$. Then $C_1 = x^2 + y^2 = 1$ and $C_2 = (x-1)^2 + y^2 = 1$. We can only have two points on the affine chart but four on $\mathbb{P}^2$ since we have points at infinity. In $\mathbb{P}^2$

$$C_1 := \{x^2 + y^2 = z^2\}$$
$$C_2 := \{(x-z)^2 + y^2 = z^2\}$$

On the affine chart, we have $x^2 = x^2 + 2xz + z^2$ thus $x = \frac{z}{2}$. And when $z = 1$, then $x = \frac{1}{2}, y = \pm\frac{\sqrt{3}}{2}$.

5.7 Dimension of Divisors

Let X be a smooth projective variety and D is a divisor in X. A Vector space $\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + D \geqslant 0\}$, where $\dim \ell(D)$ is finite.

Definition 5.7.1: Dimension of Divisor

The dimension of a divisor D in X is $\ell(D)$.

How can we calculate $\ell(D)$?

Theorem 5.7.1

For X, smooth projective curve, and D in X is either effective or a 0-divisor, then

$$\ell(D) \leqslant \deg(D) + 1$$

If $D = \sum a_i p_i$, then $\deg(D) = \sum a_i$. This theorem in particular implies that $\ell(D)$ is a finite number. Assume $D = 0$, then $\dim(\mathcal{L}(D)) \cong \dim(\mathbb{C}[X]) \cong \dim(\mathbb{C}) = 1$. On the otherhand, $\deg(D) = 0$ so $\ell(D) = 1$. Therefore the theorem is finite for $D = 0$.

Let $X = \mathbb{P}^1$. Recall $\mathcal{L}(D) = \mathcal{L}(D')$ if $D \sim D'$. If $D \subseteq \mathbb{P}^1$, as an effective divisor, then $D \sim \deg(D) \cdot pt_\alpha$. Therefore assume $D = \deg(D)pt_\alpha$. Recall in $\mathbb{P}^n$, then $D \sim D'$ if and only if $\deg(D) - \deg(D')$. Then $\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(\mathbb{P}^1) : \operatorname{div}(f) + D \geqslant 0\}$. f is regular on $\mathbb{P}^1 \setminus pt_\alpha$ and can have a pole at pt_α at most of order $\deg(D)$. Suppose $\mathbb{P}^1 \setminus pt_\alpha = \mathbb{A}^1_t, f \in \mathbb{C}[t]$ and f can have a pole of order $\deg(D)$ at pt_α means on a chart $\mathbb{A}^1_t$ where $u = \frac{1}{t}$ $\deg(t) \leqslant \deg(D)$. Then $\mathcal{L}(D) \cong \{f \in \mathbb{C}[t] : \deg(f) \leqslant \deg(D)\} \cong \mathbb{C}^{\deg(D)+1}$. Thus $\ell(D) = \deg(D) + 1$. Note that we do not always have an equality $\ell(D) = \deg(D) + 1$.

Let X be a smooth projective curve not isomorphic to $\mathbb{P}^1$. Let $D = pt \in X$. What is $\mathcal{L}(D)$?

$$\mathcal{L}(D) = \{0\} + \{f \in \mathbb{C}(X) : \operatorname{div}(f) + pt \geqslant 0\}$$

Thus f is regular on $X \setminus pt$ and has at most a first order pole at p. We claim that f is constant.

Proof: If f has no pole, $f \in \mathbb{C}[X]$, then f is constant. Assume f has a first order pole at f. If f is not constant, then $f : X \longrightarrow \mathbb{P}^1$ is a finite map, and $\deg(f) = 1$, since f has a first order pole at p. Thus f is an isomorphism, hence $X \neq \mathbb{P}^1$. Therefore $\mathcal{L}(D) \cong \mathbb{C}$, then $\ell(D) = 1 < \deg(D) + 1 = 2$. ∎

Chapter 6

Differential Forms

6.1 Regular Differential 1-Forms

Let X be a quasi-projective variety, and we want to define a regular differential 1-form on X.

Let f be a regular function on a neighbourhood U of a point $x \in X$ such that $f \in \mathbb{C}[U]$. The differential of f at x is denoted by $dxf \in T_x^*$, which is the dual of the tangent space (cotangent space). If $X = \mathbb{A}^n$, with coordinates $t_1, \dots, t_n$, and $x \in \mathbb{A}^n$, then $T_x \cong \mathbb{C}^n$. Let $f = t_i$, be the regular function on X. Projective of $(t_1, \dots, t_n) \to t_i$. What is dxt_i? It is a linear form on $\mathbb{P}^n$, represented by $dxt_i : \mathbb{C}^n \to \mathbb{C}$, which is a linear approximation to t_i. For a function $f \in \mathbb{C}[t_1, \dots, t_n]$, dxf : linear approximation of f at x, i.e. $dxf : \mathbb{C}^n \to \mathbb{C}$, defined by $(u_1, \dots, u_n) \mapsto \sum_1^n \frac{df}{dt_i}|_x \cdot u_i$. If X is not affine, we consider an affine neighborhood of $x \in X$ and reduce to the above case.

Consider the set $\Phi(X)$: { all possible set theoretic functions $\phi : X \to \coprod_{x \in X} T_x^*$, defined by $x \mapsto \phi(x) \in T_x^*$.

Definition 6.1.1: Regular Differential 1-form

A regular 1-form on X, is an element, $\phi \in \Phi(X)$, such that for all $x \in X$, there exists a neighbourhood U of x, such that $\phi|_U = \sum_{i \in I} f_i dg_i$, where $f_i, g_i \in \mathbb{C}[U]$.

If we use the language of sheaves: a regular 1-form is a global section of the cotangent sheaf. We have $\phi|_U : U \to \coprod_x T_{U,x}^*$ defined by $x \mapsto \phi(x) \in T_{U,x}^*$. On the other hand:

$$\sum_{i \in I} f_i dg_i : U \to \coprod_x T_{U,x}^*.$$

$$x \mapsto \sum_i f_i(x) dxg_i$$

So, a 1-form ϕ locally assigns to each x, $\sum f_i(x) dxg_i$.

Notation:-

$\Omega[X]$: the set of all regular 1-form. $\Omega[X]$ is a $\mathbb{C}[X]-$module.

You can add regular 1-forms, and you can multiple a 1-form $\alpha \in \Omega[X]$ by $f \in \mathbb{C}[X]$, then $f\alpha \in \Omega[X]$. There exists a natural map

$$\begin{aligned} d : &\mathbb{C}[X] \to \Omega[X] \\ &f \mapsto (df : x \mapsto dxf) \end{aligned}$$

where d is the differential map, which satisfies $d(f+g) = df + dg$, and $d(fg) = fdg + gdf$.

Theorem 6.1.1

If X is projective, then $\Omega[X]$ is a finite dimension complex vector space.

Meaning if X is projective, then $\mathbb{C}[X] = \mathbb{C}$. Then $\mathbb{C}[X]$ module is a $\mathbb{C}$ vector space.

$$\dim \Omega[X] := h^{1,0}$$

is a hodge number. If X is a smooth projective curve, then $h^{1,0}$ is a geometric genus of X.

6.2 Rational Differential 1-Forms

Let X be quasi-projective, $f \in \mathbb{C}(X)$ rational if its regular on some open dense subset $U \subseteq X$ such that $f \in \mathbb{C}[U]$. Analogously : $\Omega(X)$ is all rational differential 1-forms on X.

The choice of U does not matter, since if $\omega \in \Omega(Y)$ and $\omega' \in \Omega(U')$ and $\omega = \omega'$ on $U \cap U'$, then $\omega = \omega'$ as rational 1-forms. $\Omega(X)$ is a $\mathbb{C}(X)$ vector space. Given $\omega, \omega' \in \Omega(X), \omega + \omega' \in \Omega(X)$. Given $f \in \mathbb{C}(X), f\omega \in \Omega(X)$. We want to know the dimension of $\Omega(X)$ as a $\mathbb{C}(X)$ vector space. Claim $\dim(\Omega(X)) = \dim(X)$.

Let $X = \mathbb{A}^n$, coordinates in $t_1, \ldots, t_n$ with regular forms: $\sum_{i \in I} f_i dg_i \in \Omega[X]$. Rational 1-forms are of the form $\sum f_i dg_i \in \Omega(X)$ with $\{dt_1, \ldots, dt_n\}$ is a basis for $\Omega(X)$. So $\dim(\Omega(X)) = \dim(X) = n$ for $X = \mathbb{A}^n$. If $X = \mathbb{P}^n$¡ then $\Omega[\mathbb{P}^n] = 0$. But $\Omega(\mathbb{P}^n)$ is an n-dimensional vector space over $\mathbb{C}(\mathbb{P}^n)$. Note $\Omega(\mathbb{P}^n) = \Omega(\mathbb{A}^n)$ and generally, there exists a natural identification on $\Omega(X) = \Omega(X')$ if $X = X'$ birationally.

6.3 Behavior Under Maps of Differential 1-Forms

Let X, Y be quasi-projective varieties. $\phi : X \to Y$ is a regular map. We claim there exists a pullback map on a 1-form $\phi^* : \Omega[Y] \to \Omega[X]$.

Let $\phi : \mathbb{A}^1_t \to \mathbb{A}^1_u$ defined by $t \mapsto u = t^k$. Then $\phi^* : \Omega[\mathbb{A}^1_u] \to \Omega[\mathbb{A}^1_t]$ defined by $f(u)du \mapsto f(t^k)d(t^k)$, where $d(t^k) = kt^{k-1}dt$.

Theorem 6.3.1

Let X, Y be two smooth projective varieties birational, then $\Omega[X] \cong \Omega[Y]$.

If X is birational to Y, then there exists $\phi : X \to Y$ a rational map. ϕ is regular on some $U \subseteq X$ as an open subset $\omega \in \Omega[Y]$. Consider $(\phi|_U)^*$ is a regular 1-form on U. So $\phi|_U)^*$ is a rational 1-form on X.

Theorem 6.3.2

If X, Y are bijective, then $(\phi|_U)^*\omega$ extends to a regular 1-form.

So $\phi^* : \Omega[Y] \to \Omega[X]$ is well defined even if ϕ is rational, but X, Y are smooth projective varieties. Consequentially, for X, Y smooth projective birational varieties, we have $\dim(\Omega[X]) = \dim(\Omega[Y])$, which are related to the hodge numbers $h^{1,0}$. Birational invariants for smooth projective varieties.

Let ω, ω' be two non-identically zero L forms rational, $(\omega, \omega' \in \Omega(X))$. Then $\operatorname{div}(\omega) \sim \operatorname{div}(\omega')$. Last time we said $\Omega(X)$ is a $\mathbb{C}(X)$ vector space of dimension t. So $\{w\}$ is a basis for $\Omega(X)$, then $\omega' = gw$ where g is a rational function in $\mathbb{C}(X)$, so $\operatorname{div}(\omega') = \operatorname{div}(g) + \operatorname{div}(\omega)$, where $\operatorname{div}(g)$ is 0 in $Cl(X)$. Hence $\operatorname{div}(\omega) = \operatorname{div}(\omega'$ in $Cl(X)$. Hence $\operatorname{div}(\omega)$ is well defined as $K_x \in Cl(X)$ called the canonical class group.

Given any divisor $D \in \operatorname{Div}(X)$, there exists an associated vector space $\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + D \geqslant 0\}$ where $\dim(\mathcal{L}(D)) = \ell(D)$. What is $\mathcal{L}(K_x)$?

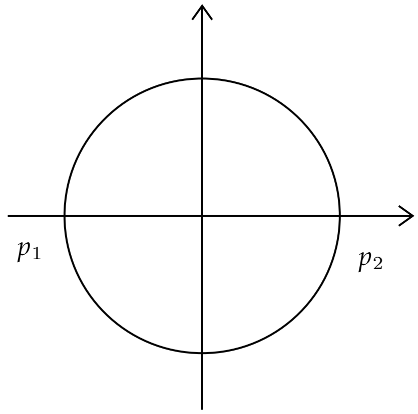
$$\mathcal{L}(K_x) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + K_x \geqslant 0\}$$

where $\operatorname{div}(f) + K_x \geqslant 0$, then $\operatorname{div}(f) + \operatorname{div}(w) \geqslant 0$. Thus $\operatorname{div}(fw) \geqslant 0$. Then fw is a regular 1-form, i.e., $fw \in \Omega[X]$. Then $\mathcal{L}(K_x) \cong \Omega[X]$.

Let $X = \mathbb{P}^1$, what is K_x? What is the degree of $K_x \in \mathbb{Z}$? Let $\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_u$, where $t = \frac{1}{u}$. Pick $\omega = dt$ to a regular 1-form on $\mathbb{A}^1_t$, then $\omega \in \mathbb{C}(\mathbb{P}^1)$ is a rational 1-form on $\mathbb{P}^1$. $\operatorname{div}(\omega)|_{\mathbb{A}^1_t} = 0$ since $\omega = 1dt$ where $\operatorname{div}(1) = 0$. $\operatorname{div}(\omega)|^1_{\mathbb{A}^u} =$?. Since $t = \frac{1}{u}$ and $dt = \frac{-1}{u^2}du$, then $\operatorname{div}(\omega)|_{\mathbb{A}^1_u} = \operatorname{div}(\frac{-1}{u^2}du) = \operatorname{div}(-u^{-2}) = -2\{pt\}$, where $pt = \{u = 0\}$. So K_x has degree -2.

6.3.1 Problems on Differential Forms

Prove that the rational differential form dx/y is regular on the affine circle X defined by $x^2 + y^2 = 1$.



If $y \neq 0$, then dx/y is regular. If $y = 0$, then $dx = 0$ at the two points, p_1, p_2 where $y = 0$. Because all tangent vectors v are vertical at p_1, p_2, then $dx : v \mapsto 0$ by definition. So $dx/y = \frac{0}{0}$ which is indeterminate. Thus we must find an alternative expression: $x^2 + y^2 = 1$. Thus we can differentiate $2xdy + 2ydy = 0$. Then $dx/y = -dy/x$, which is regular on p_1, p_2. Therefore dx/y is regular everywhere on the circle.

In the notation of the first example, we prove that $\Omega^1[X] = \mathbb{C}[X]dx/y$ which is the same of all regular 1-forms, using the notation $\Omega[X]$. We claim that $\Omega^1[X] \supseteq \mathbb{C}[X]dx/y$ is clear since by the first example, $dx.y$ is a regular 1-form, and $\Omega^1[X]$ is a $\mathbb{C}[X]$-module, so $f dx/y$ is a regular 1-form. We claim, for any $f \in \mathbb{C}[X]$, $\Omega^1[X] \subseteq \mathbb{C}[X]$: We let $\omega \in \Omega^1[X]$ be a regular 1-form. In particular, $\omega \in \Omega(X)$ since every regular 1-form is rational. If $\Omega^1(X)$ is a 1-dimensional vector space over $\mathbb{C}(X)$, then we saw previously that $\dim(\Omega^1(X)) = \dim(X) = 1$, where X is a curve. Moreover, $\{dx/y\}$ is a basis since it is not identically zero. Then $\omega = f dx/y$ where $f \in \mathbb{C}(X)$. Since ω is regular and dx/y has no zeroes, then we actually have $f \in \mathbb{C}[X]$. Hence $\omega \in \mathbb{C}[X]dx/y$.

6.4 Hypersurfaces

Let $X \subset \mathbb{P}^2$. In this section we want to describe the space of regular 1-forms $\Omega[X]$, when X is a smooth curve in $\mathbb{P}^2$, and calculate the dimension of $\Omega[X]$ and the canonical class K_x. Let $X = \{F(x_0, x_1, x_2) = 0\}$ a curve of homogenous polynomial of degree d. Assume that X is smooth. The only solution to $\frac{\partial F}{\partial x} = 0$ for all i and $F = 0$ is the point $[0 : 0 : 0] \notin \mathbb{P}^2$.

$F(x_0, x_1, x_2) = x_0^4 + x_1^4 + x_2^4$ defines a degree 4 smooth curve in $\mathbb{P}^2$. Let $U \cong \mathbb{A}^2$ be the affine chart where $x_0 \neq 0$, implying affine coordinates on U are $y_1 = \frac{x_1}{x_0}$ and $y_2 = \frac{x_2}{x_0}$. Denote $G \in \mathbb{C}[y_1, y_2]$ as a polynomial in y_1, y_2 such that $G(y_1, y_2) = F(1, y_1, y_2$. We can then define $G = 1 + y_1^4 + y_2^4$ so $\{G = 0\}$ defines an affine curve $X \cap U$.

If $\frac{\partial G}{\partial y_1} \neq 0$, then ω is regular.

Assume $\frac{\partial G}{\partial y_1} = 0$: first note that we can write

$$\omega = -\frac{1}{\frac{\partial G}{\partial y_1}} dy_2 = \frac{1}{\frac{\partial G}{\partial y_2}} dy_1$$

Equivalently,

$$dG = \frac{\partial G}{\partial y_1} dy_1 + \frac{\partial G}{\partial y_2} dy_2 = 0$$

This equation is satisfied on the curve $X \cap U$, because this equation is the equation of the tangent space to the curve. That is $G = 0$ implies $dg = 0$. Since X is smooth, we cannot have both $\frac{\partial G}{\partial y_1} = 0$ and $\frac{\partial G}{\partial y_2} = 0$. So, if $\frac{\partial G}{\partial y_1} = 0$, we must have $\frac{\partial G}{\partial y_2} \neq 0$. In this case, $\omega = \frac{1}{\frac{\partial G}{\partial y_2}} dy_1$ is regular.

Therefore ω is a nowhere zero regular 1-form $\mathrm{div}(\omega) = 0$ on $X \cap U$. That is ω has no zeroes and no poles, meaning $\omega = \frac{1}{\frac{\partial G}{\partial y_2}} dy_1$ and is regular respectively.

Consider the other affine chart $V = \{x_1 \neq 0\}$ with coordinates $z_1 = \frac{x_0}{x_1}$ and $z_2 = \frac{x_2}{x_1}$. Hence $X \cap V = \{H(z_1, z_2) = 0\}$.

We don't have to look at the third chart.

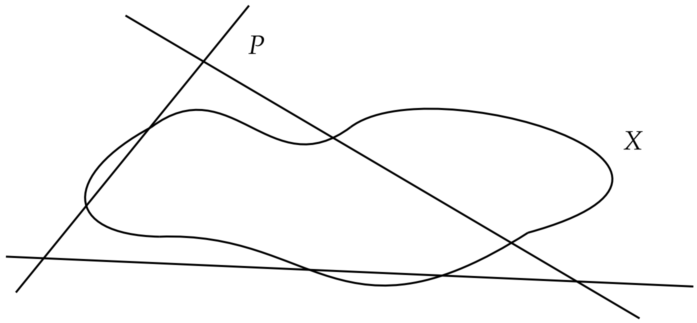


Figure 6.1: Because $\mathbb{P}^2 \setminus \{U \cup V\} = p$

We can always choose X to avoid this point. So, $\operatorname{div}(\omega) = 0$ on U and $\operatorname{div}(\omega) = (d-3)D$ on V. Thus $\operatorname{div}(\omega) = (d-3)D$ on X is a canonical divisor of class K_X.

The degree of K_X is $\deg(K_X) = d(d-3)$. If $d = 1$ meaning $X \cong \mathbb{P}^1$ line, then $\deg(K_x) = -2$. If $d = 2$ meaning $X \cong \mathbb{P}^1$ a smooth conic, then $\deg(K_X) = 2(-1) = -2$.

Chapter 7

Riemann-Roch

7.1 Riemann-Roch Theorem

Let X be a smooth, projective curve. K_X canonical divisor class. If D is a divisor, then $\mathcal{L}(D)$ vector and $\ell(D) = \dim(\mathcal{L}(D))$. $\mathcal{L}(K_x) = \Omega[X]$. And last but not least, $\dim(\mathcal{L}(K_X)) = \ell(K_X) = \dim(\Omega[X]) = g = \frac{(d-1)(d-2)}{2}$. If X is a degree d curve in $\mathbb{P}^2$.

If $d = 3$, then $g = 1$. And if $d = 4$, then $g = 3$. A smooth $g = 2$ curve will never be in $\mathbb{P}^2$ but in same $\mathbb{P}^n$.

Theorem 7.1.1

Any smooth projective curve can be embedded into $\mathbb{P}^3$.

The projectivization $\mathbb{P}(\mathcal{L}(D)) = \{D' \geqslant 0 : D' \sim D\}$. The Riemann-Roch will tell sign of dimension of this space ($\ell(D) = 1$).

$$D = \sum_i n_i p_i$$

$$\deg(D) = \sum_i n_i \in \mathbb{Z}.$$

Theorem 7.1.2 Riemann-Roch

Let X be a smooth projective curve, and D be any divisor on X. Then, $\ell(D) - \ell(K_X - D) = 1 - g + \deg(D)$.

The difference $\ell(D) - \ell(K_x - D)$ only depends on g and on the degree D. There can be many divisors of the same degree, then $\ell(D) - \ell(K_X - D)$ doesn't depend on D, but only $\deg(D)$. Note that $\ell(D)$ itself does not depend only on g and $\deg(D)$.

Let X be a curve of genus $g > 0$. Let $D_1 = 0$ as the zero divisor, and $\deg(D_1) = 0$. Then $\mathcal{L}(D_!) = \mathbb{C}[X] = c$, thus $\ell(D_1) = 1$. Let $D_2 = p - q$ for 2 distinct points p, q on X, then $\deg(D_2) = 0$. We claim that $\ell(D_2) = 0$.

Proof: $\mathcal{L}(D_2) = \{f : X \cong \mathbb{P}^1 : \deg(f) = 1\}$. Then f has at most one pole of order 1. Then $\ell(D_2) = 0$, thus $g(x) \neq 0$. Therefore, $\deg(D_1) = \deg(D_2)$, but $\ell(D_1) = 1 \neq 0 = \ell(D_2)$. ■

$$\ell(D_1) - \ell(K_X - D_1) = 1 - g$$

implies $\ell(K_X) = g$.

$$\ell(D_2) - \ell(K_X - D_2) = 1 - g$$

implies through Riemann Roch $\ell(K_X - D_2) = g - 1$

The applications of such can lead one to take $D = K_X$. Then Riemann-Roch determines $\ell(K_X) - \ell(0) = g - \ell(K_X - D) = g - 1 = 1$ point $\deg(K_X)$. Then $\deg(K_X) = 2p - 2$.

We have calculated last time for smooth curves X in $\mathbb{P}^2$. $\deg(K_X) = d(d-3)$, then $d(d-3) = 2g - 2$. Hence $g = \frac{(d-1)(d-2)}{2}$. As a corollary let L to RR imply $\deg(K_X) = 2p-2$. If $\deg(D) > 2p-2$, then $\ell(D) = 1-g+\deg(D)$. If

$\deg(D) > 2p-2$, then $\deg(K_X - D) < 0$. Since $\deg(K_X) = 2p-2$, so $\ell(K_X - D) = 0$. All efficient divisors cannot be linearly equivalent to a divisor of regular degree. Hence Riemann Roch determines $\ell(D) - \ell(\cancel{K_X} - D) = 1 - p + \deg(D)$.

Why does the question of finding rational functions $f \in \mathcal{L}(D)$ with some constraints on their zeroes and poles have to do with rational and regular 1-forms.

Theorem 7.1.3

Let X be a smooth, projective curve and $f \in \mathbb{C}(X)$ with $w \in \Omega[X]$. Then $\sum_{x \in X} \operatorname{Res}_x(fw) = 0$.

Pick a local coordinate t around x such that $fw = \left(\frac{a_k}{t^k} + \frac{a_{k-1}}{t^{k-1}} + \ldots + \frac{a_1}{t} + a_0 + \ldots\right) dt$ as the Laurant Expansion of fw. Then $\operatorname{Res}_x(fw) = a_1$ independent of the choice of local coordinate t.

Sketch of the Proof: By Cauchy's formula in complex analysis,

$$\operatorname{Res}_x(fw) = \frac{1}{2\pi i} \int fw$$

If fw has no poles intersecting in $\mathbb{R}$ is 0. Assume the poles of f are $\{p_1, \ldots, p_n\}$ and take small enough loops γ_i around each pole.

$$\sum_1^n \operatorname{Res}_{f_i}(fw) = \frac{1}{2\pi i} \left(\int_{\gamma_1} fw + \ldots + \int_{\gamma_n} fw \right)$$

where $\gamma_1 \cup \ldots \gamma_n$ is the boundary of $X \setminus \{D_1 \cup D_2 \cup \ldots \cup D_n\}$ where D_i is a disk with boundary of γ_i.

By Stoke's Theorem

$$\sum_1^n \operatorname{Res}_{p_i}(fw) = \frac{1}{2\pi i} \int_{X \setminus \{D_1 \cup \ldots \cup D_n\}} d(fw)$$

but $d(fw) = 0$ since $fw = g(t)dt$ locally and $d(g(t)dt) = d(g(t)) \wedge dt + g(t) \wedge d(dt)$, where $d(dt) = 0$. $g'(t)dt \wedge dt + g(t) \wedge 0$, where $dt \wedge dt = 0$. Hence $d(g(t)dt) = 0$. ■

Recall $\ell(D) = \ell(K_x - D) = 1 - g + \deg(D)$ such that $g = \ell(K_x) = \dim(\Omega[X])$.

Theorem 7.1.4

If $g = 0$, then $X \cong \mathbb{P}^1$.

Proof: Pick $x \in X$ and let $D = x$ as a divisor given by a single point with multiplicity 1. Then $\deg(D) = 1 > 2g - 2 = -2$. As a result, $\ell(D) = 1 - 0 + 1 = 2$. Then $\mathcal{L}(D) = \{f \in \mathbb{C}(X) : \operatorname{div}(f) + x \geqslant 0$ has $\dim 1$ and $f \in \mathbb{C}(X)$ with pole of order at most $1\} \supseteq \mathbb{C}$. Since $\ell(D) = 2$, then there exists a non-constant function $f \in \mathbb{C}(X)$ with pole of order 1. Thus $f : X \longrightarrow \mathbb{C}$ is a rational map that extends to an isomorphism $f : X \longrightarrow \mathbb{P}^1$. Therefore $X \cong \mathbb{P}^1$. ■

Theorem 7.1.5

If $g = 1$, then $K_x = 0$. That is, $g = 1$ curves are Calabi-Yau.

Proof: $\deg(K_x) = 2g - 2 = 0$, but it is not sufficient to conclude $K_x = 0$. Recall on $\mathbb{P}^1$, if there exists D, D' such that $\deg(D) = \deg(D')$, then $D \sim D'$. This is not true for $X \neq \mathbb{P}^1$.

Since $g = \ell(K_x) = 1$, there exists an effective divisor linearly equivalent to K_x, so it is 0. Effective divisor of $\deg = 0$ is the zero-divisor. Hence $K_x = 0$. ■

Theorem 7.1.6

If $g = 1$, then X is isomorphic to a cubic curve in $\mathbb{P}^1$.

Sketch of Proof: Let D be effective such that $\deg(D) > 0$. Then $\ell(D) = 1 - g + \deg(D) = \deg(D)$. Fix $p \in X$ and let $D = p$, then $\deg(D) = \ell(D) = 1$. Then $|Ell(D) = \{f \in \mathbb{C}(X)$ of pole at most 1 at $p\} \cong \mathbb{C}$. Now let $D = 2p$, then $\deg(2p) = \ell(2p) = 2 > 1$. Therefore there exists a non-constant rational function $x \in \mathbb{C}(X)$ such that $x \in \mathcal{L}(2p) \setminus \mathcal{L}(p)$ has poles of order 2.

Now let $D = 3p$, then $\deg(3p) = \ell(3p) = 3 > 2$. Thus $\mathcal{L}(D') = \{f \in \mathbb{C}(X) :$ with poles of order at most 3 at $p\}$. Since $\mathcal{L}(D)$ has poles with at most order 2 at p, then $\ell(D) = 2$, and $\ell(D') = 3$. Thus there exists a function $y \in \mathbb{C}(X)$ with poles of order 3 at p.

Consider $(x, y) : X \to \mathbb{A}^2$, a rational map which can be extended to $X \to \mathbb{P}^2$, which is an embedding. That is, it is injective. Why does $(x, y) : X \hookrightarrow \mathbb{P}^2$ define a cubic curve? Apply Riemann-Roch to the divisor $D = 6p$, then $\ell(6p) = \deg(6p) = 6$. The elements of $\mathcal{L}(6p)$ is poles of order at most 6, which are $1, x, x^2, x^3, y, y^2, xy$. Since $\ell(6p) = 6$, there exists a linear relationship with these elements, which happens to be the cubic polynomial in x, y. ■