Algebraic Geometry

Notes during my Algebraic Geometry Course

Zeroeth Edition

Algebraic Geometry

Sriaditya Vedantam
svedantam@zyphensvc.com

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Part I

Algebraic Geometry

Chapter 1

Introduction to Algebraic Geometry

Generally, people denote that Algebraic Geometry is the study of the zero set of polynomials. For example:

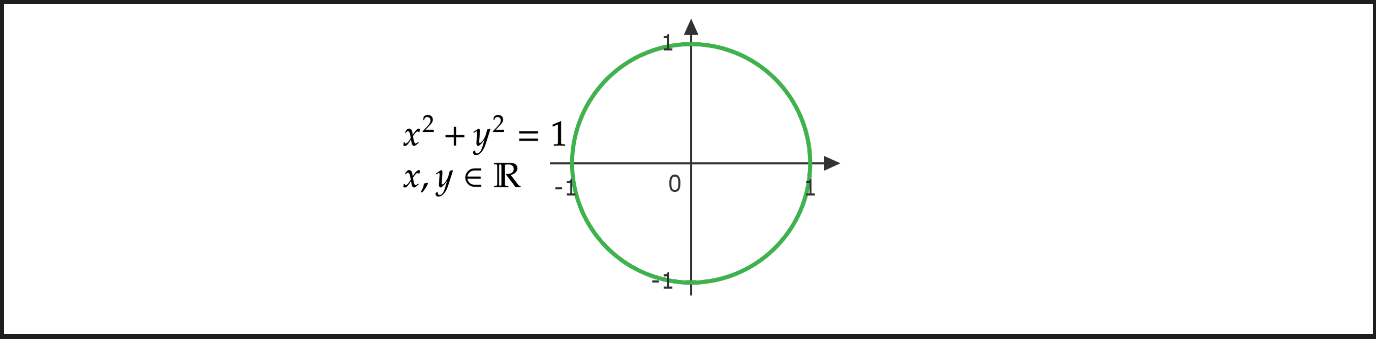


Figure 1.1: Unit Circle in the Reals

But we will be looking at solutions in specifically $\mathbb{C}$.

1.1 Algebraic Curves of the Plane

[1.1.0.1] Definition (*Affine Space*). Let $\mathbb{A}^n = \mathbb{C}^n$, which are n-tuples of complex numbers.

For example we can have $(z_1, z_2) \in \mathbb{C}^2$ where $z_1, z_2 \in \mathbb{C}$. In fact, $\mathbb{C}[x_1, \ldots, x_n]$ is the polynomial algebra in n-variables. If $f(x, y) = x^2 + y^2 - 1$, then $\{f = 0\} \subseteq \mathbb{C}^n = \mathbb{A}^n$, or $\{x^2 + y^2 - 1 = 0\}$

[1.1.0.2] Definition (*Affine Plane Curve*). In fact $f(x_1, x_2) = x_1^2 + x_2^2 - 1$, thus $\{f = 0 : x_1, x_2 \in \mathbb{C}\} \subseteq \mathbb{A}^2$, the affine plane, called an affine plane curve. Note that this curve is a $\mathbb{R} - \dim(f) = 2$, but $\mathbb{C} - \dim(f) = 1$

We often only plot the real component of affine plane curves. But if $\mathbb{A} = \mathbb{C}$, it is called an affine line, where $\mathbb{C} - \dim = 1$ and $\mathbb{R} - \dim = 2$, since we can create a complex number with two real numbers adjoined with i.This is due to the fact that if $y - x^2 = 0$, then we cannot draw a complex graph, but we can draw the real component as a parabola.

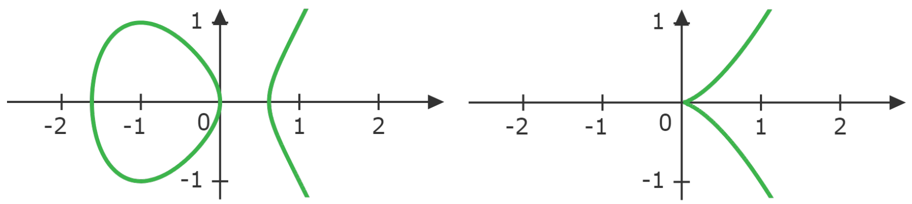


Figure 1.2: Two Graphs of Real Components

[1.1.0.3] DEFINITION (*Affine Algebraic Curve*). C is an affine curve defined by $\{f(x, y) = 0, x, y \in \mathbb{C}\} \subseteq \mathbb{A}^2$.

[1.1.0.4] DEFINITION (*Degree of C*). This is saying the same as the degree of f, which is the sum of the highest power in each variable.

$$\deg(\{xy + x + y = 0\}) = 2$$

. But all degree 2 affine plane curve polynomials are conics.

[1.1.0.5] LEMMA. Every affine conic is of the form:

1. $x^2 - y^2 = 1$ (Hyperbola);
2. $y = x^2$ (Parabola),

which are equivalent by change of coordinates (basis).

[1.1.0.6] PROBLEM. Show that ellipses can change into one of these forms.

∴

Solution.

□

Throughout this course we will be working with algebraically closed fields, for example $\mathbb{R}$ is not algebraically closed.

[1.1.0.7] Definition (*Algebraically Closed*). If every non-constant polynomial has a root in the field.

[1.1.0.8] Example. An example that $\mathbb{R}$ is not closed is shown when

$$\{x^2 + 1 = 0 : x \in \mathbb{R}\} = \varphi$$

For any field, $\mathbb{F}$, there exists $\bar{\mathbb{F}}$ that is the smallest algebraically closed field containing $\mathbb{F}$ called the algebraic closure of $\mathbb{F}$. For example $\bar{\mathbb{R}} = \mathbb{C}$.

[1.1.0.9] Proposition. Let $\mathbb{K}$ be a arbitrary field, $f, g \in \mathbb{K}[x, y]$ which are irreducible polynomials, meaning ti cannot be factored into non-constant. If g is not divisible by f, then $f(x, y) = g(x, y) = 0$ by only finitely many satisfying solutions.

We can easily understand the relationship between $\{f(x, y) = 0 : x, y \in \mathbb{C}\} \to C$, but what about the inverse direction. This is one of our goals in Algebraic Geometry. Given C, an algebraic curve which is irreducible, then we can understand its defining $\{f(x, y) = 0\}$ up to scalar.

[1.1.0.10] Definition (*Smooth Curve*). A polynomial equation that is differentiable at all points.

[1.1.0.11] Lemma. Any smooth affine conic is isomorphic to an affine plane curve.

In fact, over $\mathbb{R}$ any conic can be given by either a hyperbola, parabola, or ellipse; However, this is not true over the complexes. If we have $C \cong C'$ as affine plane curves, then we can write the following commutative diagram:

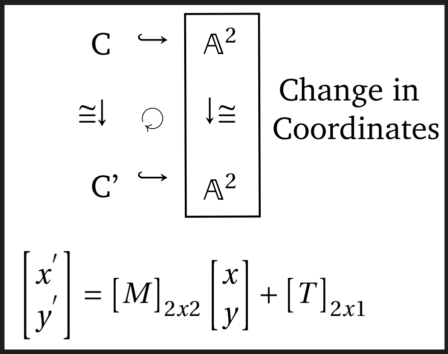


Figure 1.3: Isomorphism of Affine Curves through Change in Basis

If we take the following conic $C : \{f(x, y) = x^2 + y^2 - 1 = 0\}$, which is a circle, then we can translate it to a conic in the complexes by:

$$x \mapsto x$$

$$y \mapsto iy$$

$$x^2 + y^2 - 1 \mapsto x^2 - y^2 - 1$$

1.2 Rational Curves

Let C be an affine plane curve in $\mathbb{A}^2$.

[1.2.0.1] Definition (*Rational*). C is rational if it can be parametrized by rational functions such that there exists rational functions φ, ψ such that

$$x = \varphi(t)$$
$$y = \psi(t),$$

and $f(x, y) = 0$.

[1.2.0.2] Definition (*Rational Function*). A quotient of rational polynomials.

$$\frac{t^3 + 3t^2 + 5}{t - 1}$$

is a rational function. The domain is $\mathbb{C} \setminus \{1\}$. In fact, any complex line is rational due to parametrization.

$$f(x, y) = 2x + 3y - 5,$$

We can parameterize

$$x = t$$
$$y = \frac{-2x + 5}{3}$$

$$f(x, y) = x - 2,$$

$$x = 2$$
$$y = t$$

We can even parametrize the conic $C : \{y - x^2 = 0\}$ as

$$x = t$$
$$y = t^2$$

But simply allowing one to keep parametrizing these conics through a process of implicit isolation, we cannot have a rational function at all times. For example:

$$C : \{x^2 - y^2 = 1\}$$

$$x = t$$
$$y = \sqrt{t^2 - 1},$$

but the square root is not rational. Thus we must look at an isomorphism of this affine plane curve, by drawing a morphism between $C \cong C' : \{x^2 + y^2 - 1 = 0\}$ Note that any smooth conic is rational, due to being over any algebraically closed field.

[1.2.0.3] Lemma. If C is rational plane curve, and $C \cong C'$, then C' is rational.

[1.2.0.4] Lemma. Any affine smooth cubic that is also infinitely smooth is never rational. And any smooth projective cubic curve is never rational.

A counter example of this is if we have genus $\mathfrak{g} = \frac{(d-1)(d-2)}{2}$. Any $d = 4$ with 3 nodes contains all rational curves without a line to connect all other points.

1.3 Function Fields

Let $X := \{f(x, y) = 0 \subseteq \mathbb{A}^2$ over $\mathbb{C}\}$. And assume that X is irreducible.

[1.3.0.1] Lemma. Consider a rational function $u(x, y) = p(x, y)/q(x, y)$ and $(f \nmid q)(x, y)$. Let $u(x, y) \cong u_1(x, y) = \frac{p_1(x, y)}{q_1(x, y)}$ if and only if $pq_1 - p_1 q$ is divisible by f.

Note that $\{f = 0\} \cap \{q = 0\}$ is only finitely many points when $f \nmid q$. Thus we can simplify this lemma into a simple statement of:

[1.3.0.2] Lemma. $u(x, y) \cong v(x, y)$ if and only if $f | (pq_1 - p_1 q)$ $f \mid (pq_1 - p_1 q)$ $f | (pq_1 - p_1 q)$

[1.3.0.3] Definition (*Function Field*). Define $\mathbb{C}(X)$: field of all such rational functions.

But why do we want $f \nmid g$? It is because when $u = p/q$, it is a rational function on $\mathbb{A}^2 \setminus \{q = 0\}$. Thus we can consider the notation $\left.\frac{p}{q}\right|_X$, which is the notation for the restriction on X. Which is if and only if $f \nmid g$. Thus these forms of restrictions on rational functions can create the needs for regular functions as we are now restricting to point P or a set of finite points.

[1.3.0.4] Definition (*Regular*). If $u \in \mathbb{C}(X)$ and point $P \in X$, then u is regular at point P if u has an rational function such that $q(P) \neq 0$.

[1.3.0.5] Definition (*Regular Function*). u is regular at all points $X \setminus \{q = 0\} \cap X$, with finitely many points.

Consider $x : \{x^2 + y^2 = 1\}$ and $u = \frac{1+y}{x}$. Is u regular when $x = 0$? We in fact claim that at $P = (0, -1)$, $\lim_{(x,y)\to(0,1)} u = \frac{1-y}{x} = \infty$, thus not regular.

[1.3.0.6] Definition ($\mathbb{C}[X]$). Let X be an affine algebraic set defined by a polynomial $f(x, y) \in \mathbb{C}[x, y]$. The ring of regular functions on X is

$$\mathbb{C}[X] := \mathbb{C}[x, y]/(f(x, y)).$$

[1.3.0.7] Definition (*Integral Domain*). A ring $\mathcal{R}$ is integral if it has no zero divisors.

[1.3.0.8] Definition (*Field of Fractions*). Let $\mathcal{R}$ be an integral domain. Then

$$\text{Frac}(\mathcal{R}) := \left\{ \frac{a}{b} : a, b \in \mathcal{R},\ b \neq 0,\ \frac{a}{b} = \frac{c}{d} \iff ad = bc \right\}.$$

[1.3.0.9] Lemma. For any integral domain $\mathcal{R}$, there exists a corresponding field of fractions $\text{Frac}(\mathcal{R})$.

$$\begin{aligned} \mathcal{R} &= \mathbb{Z}, \\ \text{Frac}(\mathcal{R}) &= \mathbb{Q}, \\ \mathcal{R}_1 &= \mathbb{C}[X], \\ \text{Frac}(\mathcal{R}_1) &= \mathbb{C}(X). \end{aligned}$$

Recall that

$$\mathbb{C}(X) = \text{Frac}(\mathbb{C}[X]) = \text{Frac}\big(\mathbb{C}[x, y]/(f(x, y))\big).$$

Equivalently,

$$\mathbb{C}(X) \cong \mathbb{C}(x)[y]/(f(x, y)).$$

This means elements of $\mathbb{C}(X)$ can be viewed as rational functions in x, where y is algebraic over $\mathbb{C}(x)$ via the relation $f(x, y) = 0$.

[1.3.0.10] THEOREM. X is rational if and only if $\mathbb{C}(X) \cong \mathbb{C}(t)$ for some transcendental element t.

1.4 Rational Maps

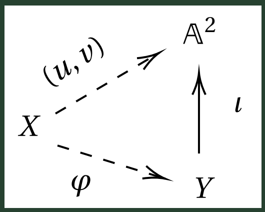
Let X, Y be irreducible plane curves. Recall that if $u = \frac{p}{q} \in \mathbb{C}(X)$, then u is defined at all but finitely many points of X. Thus we may view

$$u : X \dashrightarrow \mathbb{A}^1.$$

Given $u, v \in \mathbb{C}(X)$, we get a rational map

$$(u, v) : X \dashrightarrow \mathbb{A}^2.$$

[1.4.0.1] DEFINITION (*Rational Map*). A map $\varphi : X \dashrightarrow Y$ is rational if and only if there exist rational functions $u, v \in \mathbb{C}(X)$ such that $(u, v) : X \dashrightarrow \mathbb{A}^2$ factors through Y.



Thus

$$(u, v) = \iota \circ \varphi,$$

that is,

$$\mathrm{Im}(u, v) \subseteq Y.$$

[1.4.0.2] REMARK. If C is a rational curve, then there is a rational map

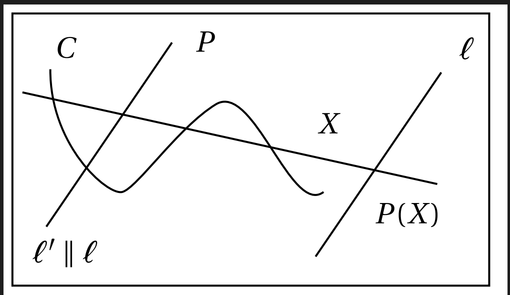
$$\mathbb{A}^1 \dashrightarrow C.$$

That is, if $C : \{f(x, y) = 0\} \subseteq \mathbb{A}^2$, then a rational parametrization has the form

$$x = u(t), \qquad y = v(t)$$

for some rational functions $u(t), v(t) \in \mathbb{C}(t)$.

Let the line $\ell \subseteq \mathbb{A}^2$. Let $P \in \mathbb{A}^2$ be a point with $P \notin \ell$.



Consider the projection map

$$\begin{aligned} P : \mathbb{A}^2 \setminus \ell' &\to \ell, \\ x &\mapsto P(x). \end{aligned}$$

Then we may consider the restriction

$$P|_C : C \dashrightarrow \ell,$$

where C is any irreducible curve.

[1.4.0.3] Definition (*Birational*). A rational map $\varphi : X \dashrightarrow Y$ is birational if and only if there exists a rational map $\psi : Y \dashrightarrow X$ satisfying

(1) $\varphi \circ \psi = \mathrm{Id}_Y$,

(2) $\psi \circ \varphi = \mathrm{Id}_X$,

whenever defined.

We call two curves birational if and only if there exists a birational map between them.

[1.4.0.4] Theorem. X is rational if and only if $\mathbb{C}(X) \cong \mathbb{C}(t)$.

∴

Proof. If X is rational, then X is birational to $\mathbb{A}^1$. A birational map induces an isomorphism of function fields, so

$$\mathbb{C}(X) \cong \mathbb{C}(\mathbb{A}^1) = \mathbb{C}(t).$$

Conversely, suppose $\mathbb{C}(X) \cong \mathbb{C}(t)$. Then $\mathbb{C}(X)$ is isomorphic to a purely transcendental extension of degree 1. By Lüroth's Theorem, the intermediate field comes from a single rational parameter. Hence X is birational to $\mathbb{A}^1$, so X is rational. □

[1.4.0.5] Theorem (*Lüroth's Theorem*). Let $K \subseteq \mathbb{C}(t)$ be a subfield with $\mathbb{C} \subseteq K$. Then there exists $r(t) \in \mathbb{C}(t)$ such that

$$K = \mathbb{C}(r(t)).$$

1.5 Singular and Non-singular Points

Let $C : \{f(x,y) = 0\} \subseteq \mathbb{A}^2$, and let $P \in C$ such that $f(P) = 0$.

[1.5.0.1] Definition (*Singular Point*). A point P is singular if and only if

$$\frac{\partial f}{\partial x}(P) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(P) = 0.$$

[1.5.0.2] Lemma. If $P = (0,0) \in C$, then $f(x,y)$ has no constant term.

[1.5.0.3] Lemma. If P is singular, then $f(x,y)$ has no linear term.

[1.5.0.4] Definition (*Smooth Curve*). If for all $P \in C$, the point P is non-singular, then C is smooth.

Smoothness is invariant under affine change of coordinates. It suffices to check that

$$x^2 - y^2 - 1 = 0 \quad \text{and} \quad x^2 - y = 0$$

are smooth.

[1.5.0.5] Theorem. An irreducible curve C has only finitely many singular points.

∴

Proof. Let $P \in C$. Define

$$C' := \left\{\frac{\partial f}{\partial x} = 0\right\}.$$

If P is singular, then $P \in C \cap C'$. From the lemma, either $C \cap C'$ is finite or C divides C', that is, $f \mid \frac{\partial f}{\partial x}$. However,

$$\deg\left(\frac{\partial f}{\partial x}\right) < \deg(f),$$

so this is impossible unless $\frac{\partial f}{\partial x} = 0$. Similarly, $\frac{\partial f}{\partial y} = 0$. If both partial derivatives are zero, then f is constant. Since we are working over $\mathbb{C}$ with *char* $= 0$, this cannot define a curve. Hence there are only finitely many singular points. □

[1.5.0.6] DEFINITION (*Multiplicity*). Let C be a curve and $P \in C$. Assume $P = (0,0)$. Then $\text{Mult}(P)$ is the smallest degree of a nonzero monomial appearing in $f(x,y)$.

$$\begin{aligned} f(x,y) &= x - y + x^2, \\ &\implies \text{Mult}(0,0) = 1. \end{aligned}$$

Recall that $\text{Mult}(P) = 1$ if and only if f contains a linear term, which implies that C is smooth at P.

$$\begin{aligned} f(x,y) &= xy, \\ &\implies \text{Mult}(0,0) = 2. \end{aligned}$$

$$\begin{aligned} f(x,y) &= y^2 - x^3, \\ &\implies \text{Mult}(0,0) = 2. \end{aligned}$$

If C is a curve of degree n, then any point on C has multiplicity at most n.

Chapter 2

Algebraic Varieties

2.1 Affine Varieties

Let $\mathbb{K} = \bar{\mathbb{K}}$, as in an algebraic closure such as $\mathbb{C}$ or $\overline{\mathbb{F}_p}$. We have already studied how

$$\mathbb{A}^n = \{(a_1, \dots, a_n) : a_i \in \mathbb{K}\}.$$

Here $\mathbb{A}^1$ is a line, and $\mathbb{A}^2$ is a plane.

[2.1.0.1] Definition (*Zariski Topology*). A closed subset of $\mathbb{A}^n$ in the Zariski topology is a set

$$Z = \{f_i(x_1, \dots, x_n) = 0 : i \in I\},$$

where $f_i \in \mathbb{K}[x_1, \dots, x_n]$ for all $i \in I$. We call this the zero set of the family $\{f_i\}_{i \in I}$.

For example, the Zariski topology in $\mathbb{A}^1$ consists of:

(1) $\varnothing$, when $f = 1$, so $\{1 = 0\} = \varnothing$.

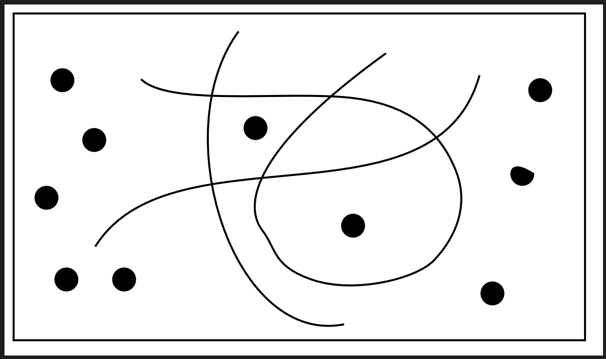
(2) Finite sets of points, since

$$f(x) = (x - \alpha_1)^{k_1} \cdots (x - \alpha_s)^{k_s}$$

has finitely many zeros.

(3) All of $\mathbb{A}^1$, when there are no equations imposed.

Whereas in $\mathbb{A}^2$:



This figure shows some points in the zero set that are of dimension 0, 1, and 2. This can be very thin if closed and dense if open.

[2.1.0.2] PROPOSITION. Affine varieties give a topology on $\mathbb{A}^n$. In particular:

(1) $\varnothing$ is closed, since $\{1=0\}=\varnothing$.

(2) $\mathbb{A}^n$ is closed, corresponding to the empty family of equations.

(3) Arbitrary intersections of closed sets are closed.

(4) Finite unions of closed sets are closed.

∴

Proof. For (3), if $Z_\alpha = \{f_{\alpha,i}=0 : i \in I_\alpha\}$, then

$$\bigcap_{\alpha\in A} Z_\alpha$$

is again the zero set of the union of all the defining equations.

For (4), if

$$Z_1 = \{f_{1j}=0\} \quad \text{and} \quad Z_2=\{f_{2k}=0\},$$

then

$$Z_1 \cup Z_2 = \{f_{1j}f_{2k}=0 \text{ for all } j,k\}.$$

□

[2.1.0.3] LEMMA. A curve is non-singular if the dimension of the tangent space is the same as the dimension of the curve.

[2.1.0.4] DEFINITION (*Ideal*). An ideal $I \subset R$ in a commutative ring is a subset such that:

1. I is a subgroup under addition, so if $i_1, i_2 \in I$, then $i_1 + i_2 \in I$.
2. If $\iota \in R$ and $i \in I$, then $\iota i \in I$.

Then I is equal to the set of finite linear combinations of some family of functions (f_i).

[2.1.0.5] Proposition. If $Z = \{f_i = 0\}$, then $Z = Z(I)$ where $I = (f_i)$.

How do we find uniqueness? Recall the following lemma.

[2.1.0.6] Lemma. Any affine variety in $\mathbb{A}^n$ can be given by finitely many equations.

∴

Proof. Let $Z = \{f_i = 0\}$. Then $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian, so every ideal is finitely generated.

The descending chain

$$\{f_1 = 0\} \supseteq \{f_1 = f_2 = 0\} \supseteq \cdots$$

corresponds to the ascending chain

$$(f_1) \subseteq (f_1, f_2) \subseteq \cdots.$$

Since the ring is Noetherian, this chain stabilizes.

Hence only finitely many equations are needed.

Uniqueness is expressed canonically by passing to the ideal $I(Z)$. □

[2.1.0.7] Definition (*Noetherian Space*). A topological space is Noetherian if and only if it satisfies the descending chain condition on closed subsets. That is, for every chain

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

of closed subsets, there exists an integer r such that

$$Y_r = Y_{r+1} = \cdots.$$

$\mathbb{A}^n$ is Noetherian. If

$$Y_1 \supseteq Y_2 \supseteq \cdots,$$

then

$$I(Y_1) \subseteq I(Y_2) \subseteq \cdots$$

is an ascending chain of ideals in $A = \mathbb{K}[x_1, \ldots, x_n]$. Since A is a Noetherian ring, this chain of ideals is stationary. Since for all i, $Y_i = Z(I(Y_i))$, the chain of Y_i is stationary as well.

[2.1.0.8] Proposition. In a Noetherian space X, every nonempty closed subset Y can be written as a finite union of irreducible closed subsets

$$Y = Y_1 \cup \cdots \cup Y_r.$$

If we also require that $Y_i \not\subseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. These are called the irreducible components of Y.

∴

Proof. Let $\mathcal{E}$ be the set of nonempty closed subsets of X which cannot be written as a finite union of irreducible closed subsets.

If $\mathcal{E}$ is nonempty, then since X is Noetherian, it contains a minimal element, say Y.

Then Y is not irreducible, by definition of $\mathcal{E}$. Thus

$$Y = Y' \cup Y'',$$

where Y' and Y'' are proper closed subsets of Y.

By minimality of Y, each of Y' and Y'' can be written as a finite union of irreducible closed subsets. This gives a decomposition of Y, a contradiction.

Hence every nonempty closed subset can be written as a finite union of irreducible closed subsets.

Now assume

$$Y = Y_1 \cup \cdots \cup Y_r = Y_1' \cup \cdots \cup Y_s',$$

where all Y_i and Y_j' are irreducible, and no one contains another.

Since

$$Y_1' \subseteq Y_1 \cup \cdots \cup Y_r,$$

and Y_1' is irreducible, we must have $Y_1' \subseteq Y_i$ for some i. Similarly, $Y_i \subseteq Y_j'$ for some j. By the non-containment assumption, this forces equality.

Proceeding by induction on the number of components gives uniqueness. □

[2.1.0.9] DEFINITION. Suppose R is a commutative ring with 1. Then

$$\sqrt{I} = \{f \in R : \exists k \geq 1 \text{ such that } f^k \in I\}.$$

[2.1.0.10] LEMMA. $\sqrt{I}$ is an ideal.

[2.1.0.11] THEOREM (*Hilbert's Nullstellensatz*). Let $\mathbb{A}$ be an ideal in $A = \mathbb{K}[x_1, \ldots, x_n]$, and let $f \in A$ be a polynomial that vanishes at all points of $Z(\mathbb{A})$. Then $f^r \in \mathbb{A}$ for some integer $r > 0$.

∴

Proof. This proof will consult many different theorems, including a paper by Daniel Allcock [All]. □

2.1.1 Hilbert's Nullstellensatz

See [All].

[2.1.1.1] THEOREM. Let $\mathbb{K}$ be a field and let $\mathbb{F}$ be a field extension that is finitely generated as a $\mathbb{K}$-algebra. Then $\mathbb{F}$ is algebraic over $\mathbb{K}$.

[2.1.1.2] DEFINITION (*Algebra*). An algebra over a field is a vector space equipped with a bilinear product.

[2.1.1.3] DEFINITION (*Bilinear*). A bilinear product satisfies distributivity and compatibility with scalar multiplication in each variable.

[2.1.1.4] THEOREM. Suppose $\mathbb{K}$ is infinite and $\mathbb{F} = \mathbb{K}(x)$. If $f_1, \ldots, f_m \in \mathbb{F}$, then the $\mathbb{K}$-algebra A generated by them is strictly smaller than $\mathbb{F}$.

∴

Proof. To see this, choose $c \in \mathbb{K}$ away from the poles of the rational functions f_i. Then no element of A can have a pole at c, so

$$\frac{1}{x-c} \notin A.$$

Thus A is strictly smaller than $\mathbb{F}$. □

[2.1.1.5] DEFINITION (*Pole*). A pole of a rational function is a value where the denominator is 0.

[2.1.1.6] THEOREM. Assume $\mathbb{F}$ is transcendental over $\mathbb{K}$. Then $\mathbb{F}$ is not finitely generated as a $\mathbb{K}$-algebra.

∴

Proof. Assume toward a contradiction that $\mathbb{F}$ is finitely generated as a $\mathbb{K}$-algebra. First suppose $\mathbb{F}$ has transcendence degree 1. Then $\mathbb{F}$ contains a subfield $\mathbb{K}(x)$ and is algebraic over $\mathbb{K}(x)$. Finite generation then implies that $\mathbb{F}$ has finite dimension over $\mathbb{K}(x)$. Choose a basis $e_1, \ldots, e_\ell$ such that

$$e_i e_j = \sum_k \frac{a_{ijk}(x)}{b_{ijk}(x)} e_k,$$

with $a_{ijk}, b_{ijk} \in \mathbb{K}[x]$. Let $f_0 = 1$ be among the generators, and write

$$f_i = \sum_j \frac{c_{ij}(x)}{d_{ij}(x)} e_j,$$

with $c_{ij}, d_{ij} \in \mathbb{K}[x]$. Then any element $a \in A$ is a $\mathbb{K}$-linear combination of products of the generators. Expanding in the basis shows that denominators come only from finitely many b's and d's. Hence there exists an irreducible polynomial in $\mathbb{K}[x]$ whose reciprocal does not lie in A. So A is strictly smaller than $\mathbb{F}$. This is a contradiction. If $\mathbb{F}$ has transcendence degree greater than 1, reduce to the transcendence degree 1 case by choosing an intermediate purely transcendental sub-extension. □

[2.1.1.7] Definition (*Transcendence Degree*). The transcendence degree of an extension field K over a field $\mathbb{F}$, denoted $\mathrm{t-deg}(K/\mathbb{F})$, is the size of a maximal algebraically independent subset over $\mathbb{F}$.

Thus $\mathbb{Q}(\pi)$ and $\mathbb{Q}(\pi, \pi^2)$ still have transcendence degree 1. This is because π^2 is algebraic over $\mathbb{Q}(\pi)$.

[2.1.1.8] Theorem (*Weak Nullstellensatz*). Let $\mathbb{K}$ be an algebraically closed field. Then every maximal ideal in the polynomial ring

$$R = \mathbb{K}[x_1, \ldots, x_n]$$

has the form

$$(x_1 - a_1, \ldots, x_n - a_n)$$

for some $a_1, \ldots, a_n \in \mathbb{K}$.

As a consequence, a family of polynomials on $\mathbb{K}^n$ with no common zero generates the unit ideal of R.

∴

Proof. If m is a maximal ideal of R, then R/m is a field finitely generated as a $\mathbb{K}$-algebra. By the previous theorem, it is algebraic over $\mathbb{K}$. Since $\mathbb{K}$ is algebraically closed, it follows that $R/m \cong \mathbb{K}$. Thus each x_i maps to some $a_i \in \mathbb{K}$ under the quotient map. So m contains the ideal

$$(x_1 - a_1, \ldots, x_n - a_n).$$

Since this is also maximal, equality holds.

For the second statement, if the given family generated a proper ideal, then that ideal would lie in some maximal ideal. By the first part, that maximal ideal corresponds to a point in $\mathbb{K}^n$ where all the polynomials vanish. This contradicts the assumption that they have no common zero. □

[2.1.1.9] Theorem (*Hilbert's Nullstellensatz*). Suppose $\mathbb{K}$ is an algebraically closed field and $g, f_1, \ldots, f_m \in R = \mathbb{K}[x_1, \ldots, x_n]$. If g vanishes on the common zero-locus of $f_1, \ldots, f_m$, then some power of g lies in the ideal generated by $f_1, \ldots, f_m$.

∴

Proof. The polynomials $f_1, \ldots, f_m$ and $x_{n+1}g - 1$ have no common zero in $\mathbb{K}^{n+1}$. By the Weak

Nullstellensatz, we can write

$$1 = p_1 f_1 + \cdots + p_m f_m + p_{m+1}(x_{n+1}g - 1),$$

where the p_i are polynomials in $x_1, \ldots, x_{n+1}$. Now apply the homomorphism

$$\mathbb{K}[x_1, \ldots, x_{n+1}] \to \mathbb{K}(x_1, \ldots, x_n)$$

given by

$$x_{n+1} \mapsto \frac{1}{g}.$$

This gives

$$1 = p_1\left(x_1, \ldots, x_n, \frac{1}{g}\right) f_1 + \cdots + p_m\left(x_1, \ldots, x_n, \frac{1}{g}\right) f_m.$$

After multiplying through by a suitable power of g to clear denominators, we obtain the result.
□

[2.1.1.10] Definition (*Locus*). A locus is the set of all points whose coordinates satisfy a given condition.

[2.1.1.11] Corollary (*Corollary of Nullstellensatz*). There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals of $\mathbb{K}[x_1, \ldots, x_n]$. This correspondence is given by

$$Y \mapsto I(Y) \quad \text{and} \quad \mathbb{A} \mapsto Z(\mathbb{A}).$$

An algebraic set is irreducible if and only if its ideal is prime.

∴

Proof. $(\Longrightarrow)$. If Y is irreducible, then $I(Y)$ is prime. If $fg \in I(Y)$, then

$$Y \subseteq Z(fg) = Z(f) \cup Z(g).$$

Thus

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g)),$$

where both are closed subsets of Y. Since Y is irreducible, we must have

$$Y \subseteq Z(f) \quad \text{or} \quad Y \subseteq Z(g).$$

Thus $f \in I(Y)$ or $g \in I(Y)$.

$(\Longleftarrow)$. Let $\mathbb{P}$ be a prime ideal, and suppose

$$Z(\mathbb{P}) = Y_1 \cup Y_2.$$

Then

$$\mathbb{P} = I(Z(\mathbb{P})) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2).$$

Since $\mathbb{P}$ is prime, it follows that $\mathbb{P} = I(Y_1)$ or $\mathbb{P} = I(Y_2)$. Hence

$$Z(\mathbb{P}) = Y_1 \quad \text{or} \quad Z(\mathbb{P}) = Y_2,$$

so $Z(\mathbb{P})$ is irreducible. □

[2.1.1.12] DEFINITION (*Prime Ideal*). A prime ideal is an ideal with the usual prime property: if $fg \in \mathbb{P}$, then $f \in \mathbb{P}$ or $g \in \mathbb{P}$.

[2.1.1.13] DEFINITION (*Irreducible Algebraic Set*). An algebraic set is irreducible if it cannot be written as the union of two proper algebraic subsets.

$\mathbb{A}^n$ is irreducible since it corresponds to the zero ideal in A, which is prime. Let f be an irreducible polynomial in $A = \mathbb{K}[x, y]$. Then f generates a prime ideal in A, since A is a unique factorization domain. So the zero set

$$Y = Z(f)$$

is irreducible. The affine curve is defined by $f(x, y) = 0$. If f has degree d, then Y is a curve of degree d.

[2.1.1.14] DEFINITION (*Integral Domain*). An integral domain is a nontrivial commutative ring in which the product of two nonzero elements is nonzero.

[2.1.1.15] DEFINITION (*Unique Factorization Domain*). A unique factorization domain is an integral domain in which every nonunit can be written as a product of irreducible elements, uniquely up to order and multiplication by units.

[2.1.1.16] LEMMA. A polynomial ring over a field is a UFD.

[2.1.1.17] DEFINITION. Hence an ideal $I \subset R$ is radical if $\sqrt{I} = I$.

A counterexample for $\mathbb{K} = \mathbb{R}$ is in $\mathbb{A}^1$:

$$Z = \{1 = 0\} = Z((1)) = Z((x^2 + 1)).$$

Thus $(x^2 + 1) \subset \mathbb{R}[x]$ is radical. Indeed, if $f^k \in (x^2 + 1)$, then $f \in (x^2 + 1)$, since $\mathbb{R}[x]$ is a UFD and $x^2 + 1$ is irreducible.

[2.1.1.18] COROLLARY. We can think of a corollary of the Nullstellensatz as

$$\{\text{radical ideals } J \subset \mathbb{K}[x_1, \dots, x_n]\} \longleftrightarrow \{\text{Zariski closed } S \subset \mathbb{A}^n_{\mathbb{K}}\}.$$

$\therefore$

Proof. □

2.2 Zariski Topology

[2.2.0.1] Lemma. An affine variety is a Zariski closed subset together with its ring of regular functions.

[2.2.0.2] Definition (*Regular Function*). Let $Z \subseteq \mathbb{A}^n$. A regular function on Z is a map $f|_Z : Z \to \mathbb{K}$ induced by a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$.

There is a quotient map

$$I(Z) \hookrightarrow \mathbb{K}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{K}[Z] := \frac{\mathbb{K}[x_1, \dots, x_n]}{I(Z)}.$$

Rings that appear this way are finitely generated $\mathbb{K}$-algebras without nilpotents.

[2.2.0.3] Definition (*Nilpotence*). An element $a \in R$ is nilpotent if $a \neq 0$ and $a^n = 0$ for some $n \geq 1$.

If $R = S/I$, then there is a quotient map

$$S \to S/I, \qquad f \mapsto \bar{f} = f \mod I.$$

Then

$$\bar{f} \neq 0 \iff f \notin I,$$

and

$$\bar{f}^n = 0 \iff f^n \in I.$$

[2.2.0.4] Definition. A regular map

$$\pi : X \to \mathbb{A}^1 = \mathbb{K}$$

is the same thing as a regular function on X.

More generally, we consider maps

$$X \to \mathbb{A}^m = \mathbb{K}^m.$$

[2.2.0.5] Definition (*Regular Map*). Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. A map

$$\pi : X \to Y$$

is regular if its coordinate functions are regular functions on X.

Equivalently, if we write

$$\vec{\pi} : X \to \mathbb{A}^m,$$

then π is regular when $\vec{\pi}(X) \subseteq Y$ and each coordinate of $\vec{\pi}$ is regular on X.

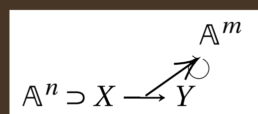
[2.2.0.6] Definition (*Isomorphism*). An isomorphism is a regular map that has a regular inverse.

[2.2.0.7] Proposition. A map $\pi : X \to Y$ is regular if and only if it induces a $\mathbb{K}$-algebra homomorphism

$$\mathbb{K}[Y] \to \mathbb{K}[X].$$

∴

Proof.



Given $\vec{\pi} : \mathbb{A}^n \to \mathbb{A}^m$, we obtain a pullback homomorphism

$$\pi^* : \mathbb{K}[y_1, \dots, y_m] \to \mathbb{K}[x_1, \dots, x_n].$$

The condition that $\vec{\pi}(X) \subseteq Y$ is equivalent to saying that $\pi^*(I(Y)) \subseteq I(X)$. Hence π^* descends to a homomorphism

$$\mathbb{K}[Y] = \frac{\mathbb{K}[y_1, \dots, y_m]}{I(Y)} \to \frac{\mathbb{K}[x_1, \dots, x_n]}{I(X)} = \mathbb{K}[X].$$

Conversely, any $\mathbb{K}$-algebra homomorphism $\mathbb{K}[Y] \to \mathbb{K}[X]$ determines a regular map $X \to Y$. □

[2.2.0.8] Corollary. There is a correspondence

$$\{\text{affine varieties}\} \leftrightarrow \{\text{finitely generated reduced } \mathbb{K}\text{-algebras}\}.$$

∴

Proof. □

[2.2.0.9] Remark. For an abstract affine variety X, the choice of an embedding $X \hookrightarrow \mathbb{A}^n$ is equivalent to the choice of algebra generators for $\mathbb{K}[X]$.

For example,

$$\frac{\mathbb{K}[x_1, x_2]}{(x_2 - x_1^2)} \cong \mathbb{K}[y].$$

[2.2.0.10] Lemma (*More about Zariski Topology*). The following hold.

(1) Closed subsets of an affine variety X correspond to radical ideals in $\mathbb{K}[X]$.

(2) The basic closed sets are $Z(f) = \{f = 0\}$, and the basic open sets are $D(f) = \{f \neq 0\}$.

(3) Regular maps are continuous.

(4) If X is irreducible, then $\mathbb{K}[X]$ has no zero divisors.

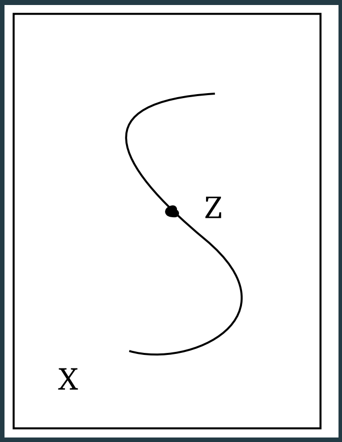
(5) $\mathbb{K}[X]$ has no nilpotents.

(6) The Zariski topology is Noetherian.

(7) A homomorphism $\mathbb{K}[Y] \to \mathbb{K}[X]$ is surjective if and only if the corresponding map $X \to Y$ identifies X with a closed subvariety of Y.

(8) A homomorphism $\mathbb{K}[Y] \to \mathbb{K}[X]$ is injective if and only if the image of the corresponding map $X \to Y$ is dense in Y.

[2.2.0.11] Remark (*Elaboration on (1)*). Suppose we have the topology X on $\mathbb{A}^n$ and a closed subset $Z \subseteq X$.



Note that $Z \subseteq X \subseteq \mathbb{A}^n$. Then we have

$$I(Z) \supseteq I(X).$$

[2.2.0.12] Definition (*Topology on a Set*). A topology on a set S consists of closed sets Z_α and open sets U_α. In the Zariski topology, the basic closed sets are

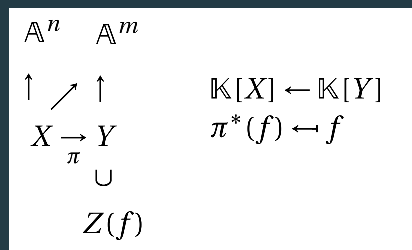
$$Z(f) = \{f = 0\},$$

and the basic open sets are

$$D(f) = \{f \neq 0\}.$$

[2.2.0.13] Remark (*Elaboration on (3)*). Recall that for a regular map $\pi : X \to Y$, and for a closed set $Z(f) \subseteq Y$, we have

$$\pi^{-1}(Z(f)) = Z(\pi^*(f)).$$



[2.2.0.14] Definition (*Irreducible*). A variety X is irreducible if it cannot be written as

$$X = Z_1 \cup Z_2$$

with $Z_1, Z_2 \subsetneq X$ both closed.

[2.2.0.15] Remark (*Elaboration on (4)*). In an ordinary topology on $\mathbb{R}$, one often studies decompositions into unions of subsets. Here, for irreducibility, it is enough to consider closed subsets.

If

$$Z(f_1) \cup Z(f_2) = X,$$

then irreducibility says that one of these must already equal X.

[2.2.0.16] Definition (*Connected*). A space is connected if it cannot be written as a disjoint union of two nonempty subsets that are both open and closed.

[2.2.0.17] Definition (*Idempotence*). An element $e \in R$ is idempotent if

$$e^2 = e.$$

[2.2.0.18] Lemma. If

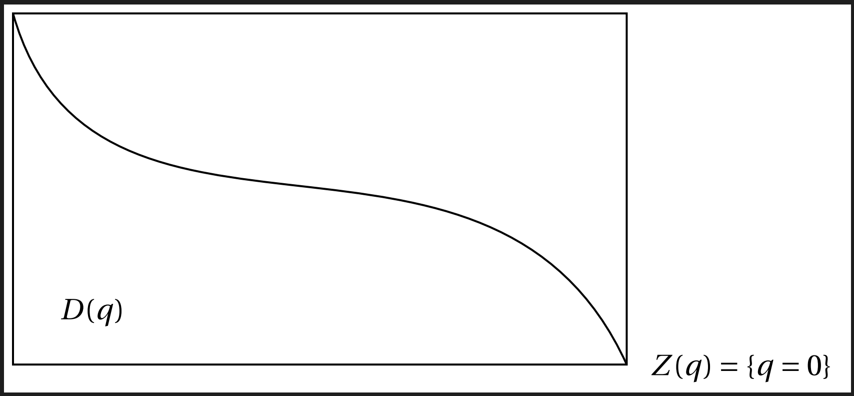
$$X = Z_1 \cup \cdots \cup Z_k$$

is a finite union of irreducible closed subsets, and no Z_i is contained in another Z_j, then the Z_i are the irreducible components of X.

[2.2.0.19] Definition (*Closure*). The closure of a subset is the intersection of all closed sets containing it.

Now assume that X is an irreducible affine variety. Then $R = \mathbb{K}[X]$ has no zero divisors. We already

have the quotient field
$$\mathbb{K}(X) = \mathrm{Frac}(R),$$
where $\varphi \in \mathbb{K}(X)$ is a rational function
$$\varphi : U \to \mathbb{K}$$
defined on some nonempty open subset $U \subseteq X$.



[2.2.0.20] LEMMA. If X is irreducible and $U_1, U_2 \subseteq X$ are nonempty open subsets, then
$$U_1 \cap U_2 \neq \varnothing.$$

∴

Proof. If $U_1 \cap U_2 = \varnothing$, then
$$X = (X \setminus U_1) \cup (X \setminus U_2),$$
where both complements are proper closed subsets of X. This contradicts irreducibility. □

2.3 Projective Varieties

Consider $U = \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2$. It is a fact that there is no natural way to give U the structure of an affine variety. On the other hand, if we remove a whole curve from an affine variety, the result can still be affine.

[2.3.0.1] DEFINITION (*Localization*).
$$\mathbb{K}[Y] := \frac{\mathbb{K}[x_1, \dots, x_n, y]}{(I(X), y - f(x_1, \dots, x_n))} \cong \mathbb{K}[X]\left[\frac{1}{f}\right].$$
More generally, if $f \in R$, then $R\left[\frac{1}{f}\right]$ denotes the localization of R at f.

[2.3.0.2] Definition ($\mathbb{P}^n$).

$$\mathbb{P}^n := \frac{\{(x_0, \dots, x_n) \in \mathbb{K}^{n+1} \setminus \{(0, \dots, 0)\}\}}{\sim},$$

where

$$(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$$

for every $\lambda \in \mathbb{K}^\times$.

Thus the points of $\mathbb{P}^n$ correspond to lines through the origin in $\mathbb{A}^{n+1}$.

[2.3.0.3] Lemma.

$$\mathbb{P}^n = U_0 \cup \cdots \cup U_n,$$

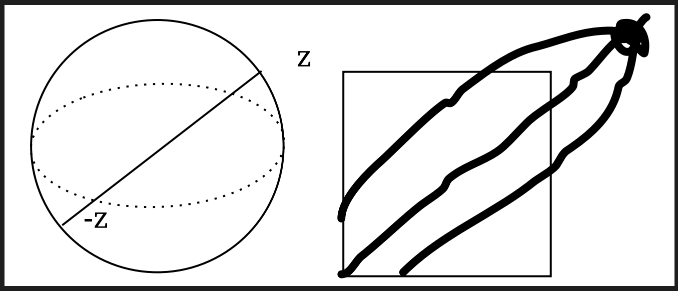
where

$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}.$$
$$\mathbb{P}^n = U_0 \cup U_1 \cup \cdots \cup U_n,$$

with each $U_i \cong \mathbb{A}^n$.

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \{\text{directions of lines}\} \cong S^2/(z \sim -z).$$



Note that for $\mathbb{P}^2_{\mathbb{C}}$, we have $\mathbb{C}-\dim(\mathbb{P}^2_{\mathbb{C}}) = 2$, while $\mathbb{R}-\dim(\mathbb{P}^2_{\mathbb{C}}) = 4$.

[2.3.0.4] Lemma. If X is a complex algebraic variety with $\mathbb{C}-\dim(X) = d$, then its underlying real manifold, when smooth, has real dimension $2d$.

Think of the following coordinates in $\mathbb{A}$ and in $\mathbb{P}$. The function x_0 makes sense on $\mathbb{A}^{n+1}$, but not on $\mathbb{P}^n$. However, the condition $x_0 = 0$ does make sense on $\mathbb{P}^n$.

[2.3.0.5] Definition (*Zero Sets in Projective Space*). If p is a homogeneous polynomial, then

$$Z_+(p)$$

denotes its well-defined zero set in projective space.

[2.3.0.6] Definition (*Homogeneous*). A polynomial is homogeneous if all of its monomials have the same degree.

[2.3.0.7] Definition (*Graded Rings*).

$$R = \bigoplus_{d=0}^{\infty} R_d = \mathbb{K}[x_0, \dots, x_n],$$

where each R_d is the space of homogeneous polynomials of degree d, and

$$R_a \cdot R_b \subseteq R_{a+b}.$$

[2.3.0.8] Definition (*Dehomogenization*). Dehomogenization is the process of passing from homogeneous coordinates to affine coordinates, for instance on the chart $x_0 \neq 0$:

$$(x_0, \dots, x_n) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

[2.3.0.9] Lemma. If p is homogeneous of degree d, then its dehomogenization on any standard affine chart has degree at most d.

2.4 Zarisky Topologies on Projective Varieties

Take a Zariski closed subset $X \subset \mathbb{P}^n$. These correspond one-to-one with radical homogeneous ideals

$$I \subset \mathbb{K}[x_1, \dots, x_{n+1}].$$

Thus we need the notion of a homogeneous ideal.

[2.4.0.1] Definition (*Homogeneous Ideal*). Take a graded ring

$$R = \bigoplus_{d=0}^{\infty} R_d$$

and an ideal $I \subset R$. We say I is homogeneous if for every $f \in I$, when we write

$$f = f_0 + f_1 + \cdots + f_d$$

with each $f_i \in R_i$, then each homogeneous piece f_i also lies in I. Equivalently, I is generated by homogeneous elements.

[2.4.0.2] Definition (*Canonical*). A subset is canonical if it is closed under scalar multiples along lines through the origin.

[2.4.0.3] Lemma. If $p(x_1,\dots,x_{n+1})$ is a homogeneous polynomial of degree d, then on the affine chart $x_0 \neq 0$ there is a corresponding polynomial

$$q(y_1,\dots,y_n) = p(1, y_1,\dots,y_n).$$

Conversely, given a polynomial $q(y_1,\dots,y_n)$ of degree at most d, the associated homogeneous polynomial is

$$x_0^d\, q\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right).$$

If we take

$$q(y_1, y_2) = y_1^2 + y_2^2 - 4,$$

then its homogeneous lift is

$$x_0^2\left(\left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 - 4\right) = x_1^2 + x_2^2 - 4x_0^2.$$

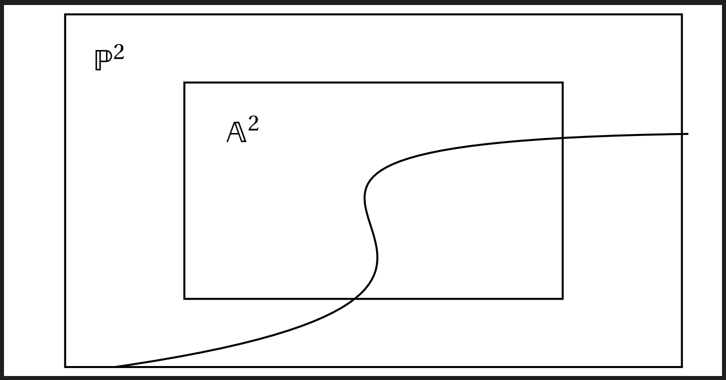
Then

$$\mathbb{P}^2 \setminus \mathbb{A}^2 = \{x_0 = 0\},$$

which is the line at infinity.

[2.4.0.4] Lemma. The Zariski topology on $\mathbb{P}^n$ restricted to the chart $\mathbb{A}^n$ is the Zariski topology on $\mathbb{A}^n$.

[2.4.0.5] Remark. Look at the following diagram.



Now the question is whether this is open or closed. In fact, the image of a projective variety under a morphism of algebraic varieties is closed in the Zariski topology. If $Y \subset \mathbb{A}^n$, then its projective closure $\bar{Y} \subset \mathbb{P}^n$ satisfies

$$I_+(\bar{Y}) = \{\hom(p) : p \in I(Y)\},$$

where $\hom(p)$ denotes the homogenization of p. We have previously defined affine varieties as Zariski closed subsets of $\mathbb{A}^n$. Now take quasi-affine varieties by starting with an affine variety and removing a closed subset. Thus they are locally closed. Similarly, we have projective and quasi-projective varieties.

[2.4.0.6] Lemma. If $U \overset{\text{open}}{\subset} X \overset{\text{closed}}{\subset} \mathbb{P}^n$ is quasi-projective, then a function $f : U \to \mathbb{K}$ is regular if for every point $V \in U$ there is a neighborhood on which f is a quotient of homogeneous polynomials of the same degree, with denominator nonzero at V.

[2.4.0.7] Lemma (*Sheaf Property*). If $Y \subset \mathbb{A}^n$ is an affine variety, then the set of regular functions agrees with the earlier definition. Moreover, the ring of regular functions on $D(f) \subset Y$ is

$$\mathbb{K}[Y]\left[\frac{1}{f}\right].$$

[2.4.0.8] Definition (*Sheaf*). A sheaf is a collection of local regular functions that are compatible on overlaps.

Maps between quasi-projective varieties are regular maps

$$f : U_1 \to U_2$$

such that for each point $v \in U_1$, there is a neighborhood on which f is given by homogeneous polynomials of the same degree:

$$f(x_1, \dots, x_{n+1}) = (p_0(x_1, \dots, x_{n+1}), \dots, p_m(x_1, \dots, x_{n+1})),$$

and at least one $p_i(v) \neq 0$.

[2.4.0.9] Corollary. Given

$$D(f) = \{f \neq 0\} \subset Y,$$

the open subset $D(f)$ is isomorphic to an affine variety.

$\therefore$

Proof. □

[2.4.0.10] Lemma. Suppose $U = X \setminus Z \subset \mathbb{P}^n$ is irreducible. Then U is not the union of two proper closed subsets. Equivalently, if $W_1, W_2 \subset U$ are nonempty open subsets, then

$$W_1 \cap W_2 \neq \varnothing.$$

The rational functions on U form a field, denoted $\mathbb{K}(U)$.

[2.4.0.11] Lemma. There is a correspondence

$$\{\text{affine varieties over } \mathbb{K}\}/\text{isomorphism} \leftrightarrow \{\text{finitely generated } \mathbb{K}\text{-algebras without nilpotents}\}.$$

Similarly,

$$\{\text{quasi-projective irreducible varieties}\}/\text{birational isomorphism} \leftrightarrow \{\text{finitely generated fields } K/\mathbb{K}\}.$$

[2.4.0.12] Definition (*Rational Map*). A rational map is a regular map defined on a nonempty open subset.

A famous example is the map

$$\mathbb{P}^1 \to \mathbb{P}^3$$

defined by

$$(x_0, x_1) \mapsto (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3).$$

Chapter 3

Algebraic Maps

3.1 Regular Maps

Let X be a quasi-projective variety with coordinates $[x_0, \dots, x_n]$.

[3.1.0.1] DEFINITION (*Regular Function*). A regular function on X is a degree 0 rational homogeneous function.

Assume $\mathbb{A}^1 \subseteq \mathbb{P}^1$ where $\mathbb{A}^1 = \{x_0 \neq 0\}$. We already know that $\mathbb{A}^1$ has coordinate ring $\mathbb{C}[t]$. We claim that $\mathbb{C}[t]$ corresponds to degree 0 rational homogeneous functions. Regular functions are regular maps to $\mathbb{A}^1$.

[3.1.0.2] LEMMA. A regular map between quasi-projective varieties X, Y is a map f such that for every point $x \in X$, there exists an open neighbourhood U of x in the Zariski topology, and an affine chart $\mathbb{A}^m \subseteq \mathbb{P}^m$ containing $f(x)$ with $f(U) \subseteq \mathbb{A}^m$, such that $f|_U : U \to \mathbb{A}^m$ is regular.

[3.1.0.3] EXAMPLE (*The Veronese Embedding*). Fix $n, d \in \mathbb{N}$. The degree d Veronese embedding of $\mathbb{P}^n$ is given by a map $f : \mathbb{P}^n \to \mathbb{P}^m$. We claim that

$$m + 1 = \binom{n+d}{d}.$$

[3.1.0.4] EXAMPLE (*The Cremona Map*). A map $\mathbb{P}^2 \to \mathbb{P}^2$ is defined by

$$[x_0 : x_1 : x_2] \longmapsto [x_1 x_2 : x_0 x_2 : x_0 x_1].$$

This is a rational map. It is regular on

$$\mathbb{P}^2 \setminus \{[1:0:0], [0:1:0], [0:0:1]\}.$$

[3.1.0.5] LEMMA. If X and Y are birational, then $\mathbb{C}(X) \cong \mathbb{C}(Y)$. The converse is also true.

3.2 Finite Maps

[3.2.0.1] PROPOSITION. Let X, Y be affine varieties, and let $f : X \to Y$ be regular such that $f(X)$ is dense in Y. Then $f^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is injective.

∴

Proof. Given f and f^*, assume $f^*(g_1) = f^*(g_2)$ for $g_1, g_2 \in \mathbb{C}[Y]$. Then $g_1 \circ f = g_2 \circ f$ on X. Thus g_1 and g_2 agree on the dense subset $f(X) \subseteq Y$. Hence $g_1 = g_2$ on Y. □

[3.2.0.2] DEFINITION (*Finite Map*). Let $f : X \to Y$ be a regular map such that $f(X) \subseteq Y$ is dense. We call f a finite map if $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$.

[3.2.0.3] DEFINITION (*Integral over a ring*). We say B is integral over A, where A, B are rings and $A \subseteq B$, if for all $b \in B$, there exists an equation

$$b^k + a_1 b^{k-1} + \ldots + a_k = 0$$

for some $k \in \mathbb{N}$, with $a_i \in A$.

Let $X := \{y^2 = x\} \subseteq \mathbb{A}^2$, and let $f : X \to \mathbb{A}^1$ be the projection map onto the x-coordinate. We claim that $\mathbb{C}[x, y]/(y^2 - x) = \mathbb{C}[X]$ is integral over $\mathbb{C}[x] = \mathbb{C}[\mathbb{A}^1]$. We want to show that each element of $\mathbb{C}[X]$ is integral over $\mathbb{C}[x]$, or at least check this on generators.

[3.2.0.4] PROPOSITION. For $X := \{y^2 = x\} \subseteq \mathbb{A}^2$, the coordinate ring $\mathbb{C}[X] = \mathbb{C}[x, y]/(y^2 - x)$ is integral over $\mathbb{C}[x]$.

∴

Proof. If $b = x$, then x satisfies the monic polynomial

$$t - x = 0$$

over $\mathbb{C}[x]$. If $b = y$, then y satisfies

$$y^2 - x = 0.$$

Thus the generators x and y are integral over $\mathbb{C}[x]$. Hence $\mathbb{C}[X]$ is integral over $\mathbb{C}[x]$. □

[3.2.0.5] DEFINITION (*Properties of finite maps*).

1. If $f : X \to Y$ is finite, then any $y \in f(X)$ has finitely many preimages. If Y is irreducible,

the degree is the generic number of preimages.

2. If f is finite, then f is surjective onto its image, and in the affine irreducible setting one usually studies it as a surjective map onto Y.

3. If f is finite and $C \subseteq X$ is closed, then $f(C)$ is closed.

3.3 Jouanolou's Trick

[3.3.0.1] THEOREM (*Jouanolou's Trick*). There exists an affine variety X together with a regular map

$$X \to \mathbb{P}^n$$

such that all fibers are affine varieties.

∴

Proof.

[3.3.0.2] DEFINITION (*Fiber*). The fiber over a point p is the preimage $f^{-1}(p)$.

We construct the example for $\mathbb{P}^1$. Recall that $\mathbb{P}^1$ parametrizes lines through the origin in $\mathbb{A}^2$. Fix the line $l := \{y = 0\} \subset \mathbb{A}^2$. Consider linear maps $M : \mathbb{A}^2 \to \mathbb{A}^2$ with image equal to a line. Such maps are represented by 2×2 matrices of rank 1. For example,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{gives} \quad M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

[3.3.0.3] DEFINITION (*Rank*). The rank is the dimension of the image, equivalently the number of linearly independent columns.

Assume $M^2 = M$. Then M is a projection, and we obtain a decomposition

$$\mathbb{A}^2 \cong \ker(M) \oplus \operatorname{Im}(M).$$

The eigenvalues of M satisfy

$$x^2 - x = 0,$$

so they are 0 and 1. The 0-eigenspace is $\ker(M)$ and the 1-eigenspace is $\operatorname{Im}(M)$. Now define X to be the set of 2×2 matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$M^2 = M \quad \text{and} \quad \operatorname{rank}(M) = 1.$$

We view X as a subset of $\mathbb{A}^4$ with coordinates (a, b, c, d). The condition $M^2 = M$ gives the polynomial equations

$$a^2 + bc = a,$$
$$ab + bd = b,$$
$$ac + dc = c,$$
$$bc + d^2 = d.$$

To ensure $\operatorname{rank}(M) = 1$, we impose

$$\det(M) = ad - bc = 0,$$

and exclude the zero matrix $M = 0$. Thus X is a quasi-affine variety (an open subset of an affine variety). Now define a map

$$\pi : X \to \mathbb{P}^1$$

by

$$M \longmapsto \operatorname{Im}(M).$$

Since $\operatorname{Im}(M)$ is a 1-dimensional subspace of $\mathbb{A}^2$, it determines a point in $\mathbb{P}^1$. This map is regular. For a fixed line $L \subset \mathbb{A}^2$, the fiber

$$\pi^{-1}(L)$$

consists of all rank 1 idempotent matrices with image L. Such matrices are determined by a choice of complementary kernel, and this parameter space is affine. Hence all fibers are affine varieties. This constructs an affine variety X mapping to $\mathbb{P}^1$ with affine fibers, which is the desired statement. □

3.4 Local Ring of a Variety at a Point

Let X be a quasi-projective variety, and let $x \in X$. What are the properties of X near x?

[3.4.0.1] THEOREM. In a quasi-projective variety, every point has a neighborhood isomorphic to an affine variety.

Note that affine subvarieties are Zariski closed in $\mathbb{A}^n$, but when embedded in projective space, affine charts are Zariski open.

[3.4.0.2] DEFINITION (*Local Ring of X at x*). Assume X is affine. Our goal is to define $\mathcal{O}_{X,x}$, the local ring of X at x. This is a special case of localization in commutative algebra. Let A be a commutative ring with 1, and let $p \subset A$ be a prime ideal. Define

$$A_p := \left\{ \frac{f}{g} : f, g \in A,\ g \notin p \right\},$$

with equivalence relation

$$\frac{f}{g} = \frac{f'}{g'} \iff \exists h \notin p \text{ such that } h(fg' - f'g) = 0.$$

Now take $A = \mathbb{C}[X]$ and $p = m_x$, the maximal ideal of functions vanishing at x. Then

$$\mathcal{O}_{X,x} := A_p = \mathbb{C}[X]_{m_x}.$$

There is a natural map

$$\varphi : A \to A_p, \quad a \mapsto \frac{a}{1}.$$

Thus we write (a, b) as $\frac{a}{b}$. Moreover, there is a unique maximal ideal

$$m = \left\{ \frac{f}{g} \in A_p : f \in p \right\}.$$

[3.4.0.3] Definition (*Local Ring*). A ring with a unique maximal ideal is called a local ring.

[3.4.0.4] Definition (*Alternative Description of $\mathcal{O}_{X,x}$*). The local ring can also be described as the stalk of the structure sheaf. Let $\mathcal{O}_X$ be the structure sheaf, where $\mathcal{O}_X(U)$ is the ring of regular functions on U. For example,

$$\mathcal{O}_X(X) = \mathbb{C}[X].$$

Then

$$\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U) \cong A_p.$$

[3.4.0.5] Remark. If $p = m_x$, then p is the maximal ideal in A, while

$$m = \left\{ \frac{f}{g} \in A_p : f \in p \right\}$$

is the maximal ideal in A_p.

[3.4.0.6] Example. Let $X = \mathbb{A}^1$. Then

$$\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t] = A.$$

Let $x = 0 \in \mathbb{A}^1$. Then

$$A_0 = \left\{ \frac{f}{g} : g(0) \neq 0 \right\}.$$

For example,

$$\frac{1}{1-t} \in A_0, \quad \frac{1}{t} \notin A_0.$$

The maximal ideal is

$$m_x = \left\{ \frac{f}{g} \in A_0 : f(0) = 0 \right\}.$$

Thus

$$\frac{1}{1-t} \notin m_x, \quad \frac{t^3}{1-t} \in m_x.$$

Chapter 4

Geometric Points

4.1 Tangent Space

Let X be an affine variety with $x \in X$. Our goal is to define the tangent space at x. We will give two definitions: the concrete and the intrinsic.

4.1.1 Concrete Definition

Assume $X \subseteq \mathbb{A}^n$ and, without loss of generality, let $x = (0, \ldots, 0) \in \mathbb{A}^n$. Our idea is that the tangent space at x, denoted T_x, is the union of all lines tangent to X at x. A tangent space exists at singular points, not just smooth points. Different notions of tangency may not agree. Thus calculus alone is insufficient, and we instead use intersection multiplicity.

[4.1.1.1] Definition. A line is tangent at a point x if it has intersection multiplicity ≥ 2 at x.

[4.1.1.2] Proposition. Let ℓ be a line in $\mathbb{A}^n$ passing through $x = (0, \ldots, 0)$. Pick a point $a = (a_1, \ldots, a_n) \in \ell$. Then all points on ℓ are of the form ta for $t \in \mathbb{C}$. Let $X = \{f_1 = \cdots = f_m = 0\} \subseteq \mathbb{A}^n$. For each i, the restriction $f_i|_\ell$ is a polynomial in t. Since $f_i(0) = 0$, we may write

$$f_i|_\ell = t^{k_i} g_i(t), \quad g_i(0) \neq 0.$$

[4.1.1.3] Definition (*Multiplicity of x as a root*). The integer $k_i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is called the multiplicity (order of vanishing) of $f_i|_\ell$ at 0. If $f_i|_\ell \equiv 0$, then $k_i = \infty$.

[4.1.1.4] Definition (*Intersection multiplicity*).

$$\mu := \min_{1 \le i \le m} \{k_i\}.$$

[4.1.1.5] Definition (*Line of tangency*). A line ℓ is tangent at x if $\mu \ge 2$.

[4.1.1.6] Definition (*Tangent space*).

$$T_x := \bigcup_{\ell \text{ tangent at } x} \ell.$$

[4.1.1.7] Example.

$$X = \{y = x^2\} \subseteq \mathbb{A}^2.$$

Pick ℓ with parameterization $(a_1 t, a_2 t)$.

$$f(x, y) = x^2 - y,$$

$$f|_\ell = (a_1 t)^2 - a_2 t = a_1^2 t^2 - a_2 t.$$

Then $\mu \ge 2 \iff a_2 = 0$. Thus the tangent line is $\{y = 0\}$.

[4.1.1.8] Example.

$$X = \{y(y - x^2) = 0\} \subseteq \mathbb{A}^2.$$

Let ℓ be parameterized by $(a_1 t, a_2 t)$.

$$f|_\ell = a_2 t(a_2 t - (a_1 t)^2)$$
$$= a_2^2 t^2 - a_1 a_2 t^3.$$

In all cases, $\mu \ge 2$. Hence every line through $(0,0)$ is tangent, and

$$T_{(0,0)} = \mathbb{A}^2.$$

[4.1.1.9] Example.

$$X = \{y = ax\} \subseteq \mathbb{A}^2.$$

$$f(x, y) = y - ax,$$

$$f(a_1 t, a_2 t) = (a_2 - a a_1) t.$$

We have $\mu = \infty$ only when $a_2 = a a_1$. Thus the tangent line is $\{y = ax\}$.

4.1.2 Intrinsic Definition

Let $\mathcal{O}_{X,x}$ be the local ring at x with maximal ideal $\mathfrak{m}_x$. Define

$$\mathfrak{m}_x^2 := \left\{\sum a_i b_i : a_i, b_i \in \mathfrak{m}_x\right\}.$$

[4.1.2.1] DEFINITION (*Cotangent space*). The cotangent space is

$$\mathfrak{m}_x/\mathfrak{m}_x^2.$$

[4.1.2.2] THEOREM.

$$T_x \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee.$$

[4.1.2.3] DEFINITION (*Lift*). Given morphisms $f : X \to Y$ and $g : Z \to Y$, we say f lifts through Z if there exists $h : X \to Z$ such that $f = g \circ h$.

[4.1.2.4] REMARK. The cotangent space is a vector space over the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$, and its dual is the tangent space.

[4.1.2.5] PROPOSITION. Let $X \subseteq \mathbb{A}^n$ be affine with defining equations $f_1, \ldots, f_m$. Then

$$T_x = \{a \in \mathbb{A}^n : d_x f_i(a) = 0,\ \forall i\}.$$

[4.1.2.6] EXAMPLE. If $X = \mathbb{A}^1$, then $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t]$, $\mathfrak{m}_0 = (t)$, and $\mathfrak{m}_0^2 = (t^2)$. Thus

$$\mathfrak{m}_0/\mathfrak{m}_0^2 \cong \mathbb{C},$$

so $T_0\mathbb{A}^1 \cong \mathbb{C}$.

4.2 Dimension and Singular/Nonsingular Points

Assume X is irreducible. Consider $\mathbb{C}(X)$.

[4.2.0.1] DEFINITION (*Transcendence Degree*). The dimension of X is the transcendence degree of the field extension $\mathbb{C}(X)$ over $\mathbb{C}$. This can also be defined as the maximum number of algebraically independent elements of $\mathbb{C}(X)$ over $\mathbb{C}$.

[4.2.0.2] DEFINITION (*Algebraically Independent Elements*). Let $f_1, \dots, f_k \in \mathbb{C}(X)$. They are algebraically independent over $\mathbb{C}$ if there does not exist a nonzero polynomial $p(x_1, \dots, x_k)$ such that $p(f_1, \dots, f_k) = 0$.

[4.2.0.3] EXAMPLE. Take $X = \mathbb{A}^1$ and $\mathbb{C}(X) = \mathbb{C}(t)$. We claim that $\mathrm{t-deg}(\mathbb{C}(t))$ over $\mathbb{C}$ is 1.

[4.2.0.4] PROPOSITION. $\mathrm{t-deg}(\mathbb{C}(t)/\mathbb{C}) = 1$.

∴

Proof. t is algebraically independent over $\mathbb{C}$. Thus $\mathrm{t-deg}(\mathbb{C}(t)) \geq 1$.

We show equality by proving that for any $f \in \mathbb{C}(t)$, the set $\{f, t\}$ is algebraically dependent.

Write $f = \dfrac{p_1(t)}{p_2(t)}$ with $p_1, p_2 \in \mathbb{C}[t]$ and $p_2 \neq 0$.

Define

$$p(x_1, x_2) := p_2(x_1)x_2 - p_1(x_1).$$

Then

$$p(t, f) = p_2(t)f - p_1(t) = 0.$$

Hence $\{f, t\}$ is algebraically dependent, and $\mathrm{t-deg}(\mathbb{C}(t)) = 1$. □

Similarly, $\mathrm{t-deg}(\mathbb{C}(\mathbb{A}^n)) = n$. Recall that X, Y are birational if and only if $\mathbb{C}(X) \cong \mathbb{C}(Y)$. Thus $\dim(\mathbb{P}^n) = n$ since $\mathbb{C}(\mathbb{P}^n) = \mathbb{C}(t_1, \dots, t_n)$.

[4.2.0.5] THEOREM. Let $f(x_1, \dots, x_n)$ be a nonzero irreducible polynomial. Then $\dim(X) = n - 1$, where $X = V(f) \subseteq \mathbb{A}^n$.

[4.2.0.6] LEMMA. The dimension of a hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ is $n - 1$.

[4.2.0.7] COROLLARY. If X is not irreducible and $X = \bigcup X_i$ is the decomposition into irreducible components, then

$$\dim(X) = \max_i \{\dim(X_i)\}.$$

[4.2.0.8] DEFINITION (*Local Dimension*).

$$\dim_x(X) := \max\{\dim(X_i) : x \in X_i\}.$$

[4.2.0.9] THEOREM. For all $x \in X$, we have

$$\dim(T_x) \geq \dim_x(X).$$

[4.2.0.10] DEFINITION (*Nonsingular Points*). A point $x \in X$ is nonsingular (smooth) if and only if

$$\dim(T_x) = \dim_x(X).$$

4.3 Normal Varieties

4.3.1 Integrally Normal

Let

$$A = \mathbb{C}[x, y]/(y^2 - x^2 - x^3).$$

We claim that A is not integrally closed. Recall that if $K = \mathrm{Frac}(A)$, then

$$t := \frac{y}{x} \in K.$$

Then

$$t^2 = \frac{y^2}{x^2} = \frac{x^2 + x^3}{x^2} = 1 + x \in A.$$

Thus

$$t^2 - 1 - x = 0,$$

which is a monic polynomial equation over A satisfied by t. Hence t is integral over A. But $t \notin A$. Therefore A is not integrally closed. If $p/q \in \mathbb{C}(X)$, assume that p/q is integral over $\mathbb{C}[X]$. Then there exists a monic equation

$$\left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + \cdots + a_n = 0$$

with $a_i \in \mathbb{C}[X]$. Multiplying through by q^n, we get

$$p^n + a_1 p^{n-1} q + \cdots + a_n q^n = 0.$$

Hence

$$p^n = -\left(a_1 p^{n-1} q + \cdots + a_n q^n\right).$$

So q divides p^n in $\mathbb{C}[X]$. If $\mathbb{C}[X]$ is integrally closed, this forces $p/q \in \mathbb{C}[X]$.

[4.3.1.1] DEFINITION (*Normal*). An irreducible affine variety X is normal if and only if $\mathbb{C}[X]$ is integrally closed.

Generally, an irreducible quasi-projective variety X is normal if and only if every point has a normal affine neighborhood.

[4.3.1.2] LEMMA. If X is an irreducible quasi-projective variety, then X is normal if and only if $\mathcal{O}_x$ is integrally closed for all $x \in X$.

[4.3.1.3] LEMMA. X is normal if and only if every finite birational morphism to X is an isomorphism.

∴

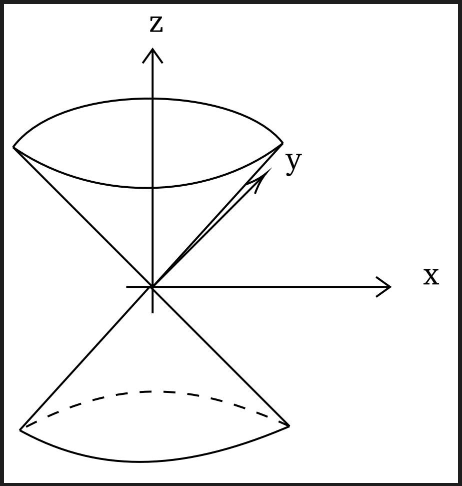
Proof. □

[4.3.1.4] THEOREM. An irreducible curve is normal if and only if it is smooth.

[4.3.1.5] THEOREM. If X is an irreducible quasi-projective variety and is smooth, then it is normal.

[4.3.1.6] THEOREM. If X is an irreducible quasi-projective variety and is normal, then the set of singular points in X has codimension at least 2.

$$\{x^2 + y^2 = z^2\} =: S \subseteq \mathbb{C}^3.$$



We claim that S is normal.

[4.3.1.7] EXAMPLE. Let

$$S = \{x^2 + y^2 = z^2\} \subseteq \mathbb{C}^3.$$

Then S is normal. With $u, v \in \mathbb{C}(x, y)$, every element of $\mathbb{C}(S)$ has the form $u + vz$. If $\alpha = u + vz \in \mathbb{C}(S)$ is integral over $\mathbb{C}[S]$, then it is also integral over $\mathbb{C}[x, y]$ since $\mathbb{C}[S]$ is finite over $\mathbb{C}[x, y]$.

The conjugate of α over $\mathbb{C}(x,y)$ is $u - vz$. So the minimal polynomial of α over $\mathbb{C}(x,y)$ is

$$T^2 - 2uT + u^2 - (x^2+y^2)v^2.$$

If α is integral over $\mathbb{C}[x,y]$, then the coefficients of this polynomial are integral over $\mathbb{C}[x,y]$. Since $\mathbb{C}[x,y]$ is integrally closed, it follows that

$$2u \in \mathbb{C}[x,y],$$

hence

$$u \in \mathbb{C}[x,y].$$

Also

$$u^2 - (x^2+y^2)v^2 \in \mathbb{C}[x,y].$$

Since

$$x^2 + y^2 = (x+iy)(x-iy),$$

and these factors are irreducible and coprime in $\mathbb{C}[x,y]$, it follows that $v \in \mathbb{C}[x,y]$. Therefore $\alpha \in \mathbb{C}[S]$. So $\mathbb{C}[S]$ is integrally closed, and hence S is normal.

4.3.2 Normalization

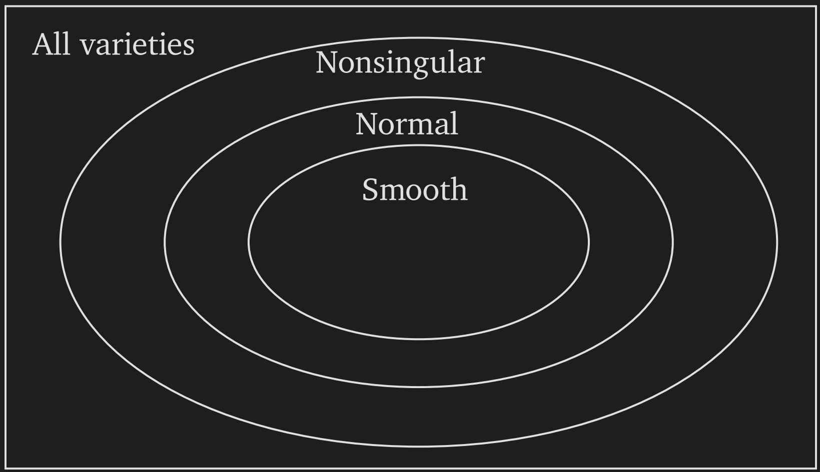


Figure 4.1: From Mumford's *Red Book of Varieties and Schemes*.

[4.3.2.1] PROPOSITION. Let $Y = \operatorname{Spec}(S)$ be normal. Let $R \subset S$ be a finitely generated subring such that S is finite over R. Then $X := \operatorname{Spec}(R)$ need not be normal.

Let $S = \mathbb{C}[x,y]$, so $\operatorname{Spec} S = \mathbb{A}^2$. Take

$$R = \mathbb{C} + I,$$

where I is the ideal of the two points $(0,0)$ and $(1,0)$. Then

$$0 \to R \to S \to S/R \to 0,$$

and

$$S/R \cong \mathbb{C} \oplus \mathbb{C}.$$

[4.3.2.2] Definition (*Normalization*). Let X be an irreducible quasi-projective variety. A normalization of X is a normal irreducible variety denoted by X^ν together with a morphism

$$\pi : X^\nu \to X$$

which is finite and birational. If X is not irreducible, then write $X = \bigcup X_i$ as the union of irreducible components and define

$$X^\nu = \bigsqcup X_i^\nu.$$

[4.3.2.3] Theorem (*Geometric Noether Normalization Theorem*). Let X be an affine variety of dimension n. Then there exists a finite morphism

$$\pi : X \to \mathbb{A}^n$$

[4.3.2.4] Definition ($V(S)$). For every subset $S \subset R$, we have

$$\begin{aligned} V(S) &= \{x \in \operatorname{Spec}(R) : f(x) = 0 \text{ for all } f \in S\} \\ &= \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal and } \mathfrak{p} \supseteq S\}. \end{aligned}$$

[4.3.2.5] Theorem. Let R be an integral domain finitely generated over $\mathbb{C}$, and let $P \subset R$ be a prime ideal. Then

$$\dim(R/P) = \operatorname{t-deg}_{\mathbb{C}} \operatorname{Frac}(R/P).$$

∴

Proof. We can reduce this proof to the case $P = \sqrt{(f)}$, or geometrically, the case $Z = V((f))$. Suppose we have the decomposition

$$\sqrt{(f)} = P \cap P_1' \cap \ldots \cap P_t'$$

in R. If $Z_i' = V(P_i')$, then $Z, Z_1', \ldots, Z_t'$ are the components of $V((f))$. Pick an affine open $U_0 \subset X$ such that

$$\begin{aligned} U_0 \cap Z &\neq \varnothing, \\ U_0 \cap Z_i' &= \varnothing, \qquad i = 1, \ldots, t. \end{aligned}$$

Let $U_0 = X_g$, where

$$g \in P_1' \cap \ldots \cap P_t', \qquad g \notin P.$$

Then replace X by U_0, R by $R_{(g)}$, and in the new setup

$$\begin{aligned} V_{U_0}((f)) &= V_X((f)) \cap U_0 \\ &= Z \cap U_0 \end{aligned}$$

is irreducible. Hence in $R_{(g)}$, $\sqrt{(f)} = P \cdot R_{(g)}$ is prime. We can now use Noether normalization to find a morphism

$$X \xrightarrow{\pi} \mathbb{A}^n,$$

equivalently

$$R \overset{\pi^*}{\hookleftarrow} \mathbb{C}[x_1, \ldots, x_n] = S.$$

Let K be the quotient field of R and let L be the quotient field of S. Then K/L is a finite algebraic extension. Let

$$f_0 = N_{K/L}(f).$$

Then we claim $f_0 \in S$ and

$$P \cap S = \sqrt{(f_0)}.$$

If we prove this, the theorem follows. For R/P is an integral extension of $S/(S \cap P)$, so

$$\mathrm{t-deg}_{\mathbb{C}} R/P = \mathrm{t-deg}_{\mathbb{C}} S/(S \cap P).$$

But S is a UFD, so if $S \cap P$ is height one, then it is principal. Therefore $P \cap S = (f_{00})$ for some irreducible $f_{00} \in S$, and hence

$$\mathrm{t-deg}_{\mathbb{C}} S/(S \cap P) = n - 1.$$

We check first that $f_0 \in P \cap S$. Let

$$Y^n + a_1 Y^{n-1} + \ldots + a_n = 0$$

be the irreducible equation satisfied by f over the field L. Then f_0 is a power of a_n. Moreover, all the a_i are symmetric functions in the conjugates of f, therefore the a_i are elements of L integral over S. Hence $a_i \in S$. In particular $f_0 \in S$, and since

$$\begin{aligned} 0 &= a_n^{m-1}(f^n + a_1 f^{n-1} + \ldots + a_n) \\ &= f(a_n^{m-1} f^{n-1} + a_n^{m-1} a_1 f^{n-2} + \ldots + a_n^{m-1} a_{n-1}) + f_0, \end{aligned}$$

we also have $f_0 \in P$. Finally suppose $g \in P \cap S$. Then $g \in P$, hence $g^n = fh$ for some integer n and some $h \in R$. Taking norms, we find that

$$\begin{aligned} g^{n[K:L]} &= N_{K/L}(g^n) \\ &= N_{K/L}(f) N_{K/L}(h) \in (f_0), \end{aligned}$$

since $N_{K/L}(h)$ is an element of S by the reasoning used before. Therefore $g \in \sqrt{(f_0)}$, and so $P \cap S = \sqrt{(f_0)}$. □

[4.3.2.6] THEOREM. The normalization of any quasi-projective variety exists and is unique up to isomorphism compatible with π.

∴

Proof. Assume X is affine. Let

$$A := \{f \in \mathbb{C}(X) : f \text{ is integral over } \mathbb{C}[X]\}.$$

Then A is a subring of $\mathbb{C}(X)$. Thus there are no zero divisors in A, and it is finitely generated over $\mathbb{C}$ by Noether normalization. Then A is the ring of regular functions on an affine variety.

Let
$$X^\nu := \operatorname{Spec}(A).$$
We claim that X^ν is the normalization of X, with morphism
$$X^\nu \xrightarrow{\varphi} X$$
induced by $\mathbb{C}[X] \hookrightarrow A$. In order to prove this, we must check that φ is birational and finite. □

If we let
$$X := \operatorname{Spec}(\mathbb{C}[x,y]/(y^2 - x^3)),$$
then $\mathbb{C}[X]$ is not normal.

4.4 Singularities

Suppose $f : X \to Y$ is a regular map, where X, Y are quasi-projective varieties. The question that arises is: if X is irreducible, are the fibers of f irreducible, and if X is smooth, are the fibers smooth? Generally this is not true. For example,
$$\begin{aligned} \mathbb{A}^1 &\xrightarrow{f} \mathbb{A}^1 \\ x &\longmapsto x^2 \end{aligned}$$
where $f^{-1}(y)$ is not irreducible for general y.

[4.4.0.1] Theorem (*First Bertini's Theorem*). Assume X, Y are irreducible and $f(X)$ is dense in Y, that is, f is dominant. Suppose X remains irreducible over the algebraic closure $\overline{\mathbb{C}(Y)}$. Then there exists an open dense subset $U \subseteq Y$ such that for all $y \in U$, the fiber $f^{-1}(y)$ is irreducible.

If we have the previous example $\mathbb{A}^1 \to \mathbb{A}^1$, then $\mathbb{C}[Y] = \mathbb{C}[t]$, so $\mathbb{C}(Y) = \mathbb{C}(t)$, which is not algebraically closed. Since we do not have $t^{1/2}$, and even if we adjoin one root, we still will not have all $t^{1/n}$.

[4.4.0.2] Theorem.
$$\overline{\mathbb{C}(t)} = \bigcup_{n \geq 1} \mathbb{C}(t^{1/n}),$$
provided *char*0.

How can we view X as an irreducible variety over $\overline{\mathbb{C}(Y)}$? Assume X, Y are affine. Then f induces $\mathbb{C}[Y] \to \mathbb{C}[X]$ by $\varphi \mapsto \varphi \circ f$. Thus $\mathbb{C}[X]$ is a $\mathbb{C}[Y]$-algebra. Note that an algebra is a vector space equipped with a bilinear product. We have
$$\mathbb{C}[Y] \subseteq \mathbb{C}(Y) \subseteq \overline{\mathbb{C}(Y)}.$$
Then consider
$$\mathbb{C}[X] \otimes_{\mathbb{C}[Y]} \overline{\mathbb{C}(Y)}.$$

[4.4.0.3] DEFINITION (*Tensor Product*). An element of $A \otimes_C B$ satisfies the relation

$$ca \otimes b = a \otimes cb.$$

Thus if we let

$$\tilde{X} := \operatorname{Spec}\left(\mathbb{C}[X] \otimes_{\mathbb{C}[Y]} \overline{\mathbb{C}(Y)}\right),$$

then $\tilde{X}$ is affine. In the statement of First Bertini's Theorem, we have that $\tilde{X}$ is irreducible if and only if X is irreducible over $\overline{\mathbb{C}(Y)}$. Consider

$$\tilde{X} = \operatorname{Spec}\left(\mathbb{C}[X] \otimes_{\mathbb{C}[t]} \overline{\mathbb{C}(t)}\right).$$

We claim that $\tilde{X}$ is not irreducible. Since

$$\mathbb{C}[x] \cong \mathbb{C}[t][x]/(x^2 - t)$$

as a $\mathbb{C}[t]$-algebra, we get

$$\mathbb{C}[t][x]/(x^2 - t) \otimes_{\mathbb{C}[t]} \overline{\mathbb{C}(t)} \cong \overline{\mathbb{C}(t)}[x]/(x^2 - t).$$

We can now factor over $\overline{\mathbb{C}(t)}$:

$$\overline{\mathbb{C}(t)}[x]/(x^2 - t) = \overline{\mathbb{C}(t)}[x]/\left(x - \sqrt{t}\right)\left(x + \sqrt{t}\right).$$

Thus

$$\tilde{X} = \operatorname{Spec}\left(\overline{\mathbb{C}(t)}[x]/\left(x - \sqrt{t}\right)\left(x + \sqrt{t}\right)\right)$$

has two points, hence is not irreducible. The First Bertini Theorem ensures irreducibility over points $U \subseteq Y$ open and dense, but not all of Y.

[4.4.0.4] THEOREM (*Second Bertini's Theorem*). Given $f : X \to Y$ with X, Y quasi-projective, $f(X)$ dense in Y, and X nonsingular, there exists an open dense subset $U \subseteq Y$ such that for all $y \in U$, the fiber $f^{-1}(y)$ is smooth.

We can think of this theorem as the algebro-geometric analog of Sard's Theorem from differential geometry.

[4.4.0.5] THEOREM (*Sard's Theorem*). Let $f : M \to N$ be a smooth map of manifolds. Then the set of critical values of f has measure zero in N.

[4.4.0.6] DEFINITION (*Critical Points*). A point $x \in M$ is a critical point if and only if

$$d_x f : T_x M \to T_{f(x)} N$$

is not surjective.

Chapter 5

Divisors

5.1 Divisors

Suppose X is a quasi-projective irreducible variety.

[5.1.0.1] Definition (*Prime Divisor*). A prime divisor on X is an irreducible subvariety of X of codimension 1.

If $X = \mathbb{A}^1$ or $\mathbb{P}^1$, then a prime divisor is a single point. If X is a surface, then a prime divisor is a curve.

[5.1.0.2] Definition (*Weil Divisor*). A Weil divisor D on X is given by

$$D = k_1 C_1 + \ldots + k_r C_r,$$

where $k_i \in \mathbb{Z}$ and each C_i is a prime divisor.

We define

$$\mathrm{Div}(X) := \left\{ \sum k_i C_i : k_i \in \mathbb{Z},\ C_i \text{ prime divisors} \right\},$$

which is an abelian group. We define the ***support*** of D to be the union of all C_i.

[5.1.0.3] Definition (*Effective*). A divisor D is effective if $k_i \geq 0$ for all i, and not all k_i are 0.

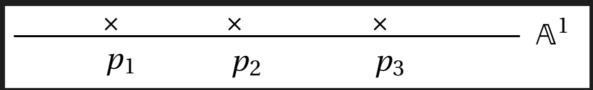


Figure 5.1: Defining points for a divisor on $\mathbb{A}^1$, for example the divisor $3p_1 - p_2 + 5p_3$.

Thus the corresponding rational function can be thought of as

$$\frac{f_1^3 f_3^5}{f_2}.$$

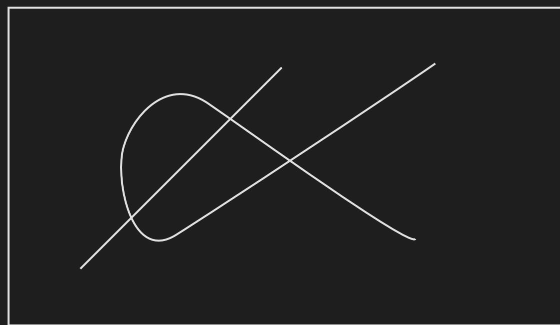
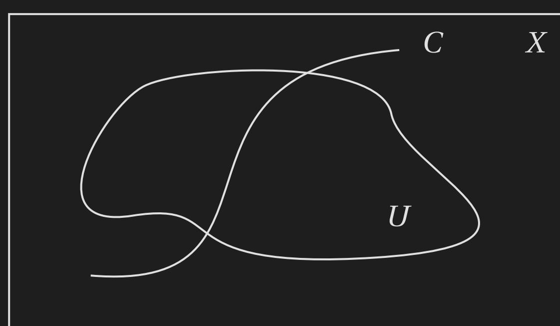


Figure 5.2: All points on the line and curve.

We state our goal. Given $f \in \mathbb{C}(X) \setminus \{0\}$, we define what $\operatorname{div}(f) \in \operatorname{Div}(X)$ is. Assume X is nonsingular in codimension 1. This means $\operatorname{Sing}(X)$ has codimension at least 2, so in particular X is normal. Given f, we define the order of vanishing $\nu_C(f)$ along a prime divisor $C \subset X$. Pick $U \subset X$ affine and dense such that $C \cap U \neq \varnothing$. Then $C \cap U$ is a prime divisor of U, and since U is normal, it is cut out by a prime element π locally.



Write

$$f|_U = \frac{g}{h} \in \mathbb{C}(U),$$

with $g, h \in \mathbb{C}[U]$. There is a unique integer $k \in \mathbb{Z}_{\geq 0}$ such that $g \in (\pi)^k$ but $g \notin (\pi)^{k+1}$. Then k is the largest integer such that $\pi^k \mid g$. We set $\nu_C(g) = k$. Define similarly $\nu_C(h)$. Finally set

$$\nu_C(f) := \nu_C(g) - \nu_C(h).$$

We remark that $\nu_C(f)$ is independent of the choice of U. If $X = \mathbb{A}^1$, then $\mathbb{C}(X) = \mathbb{C}(t)$. Suppose

$$f(t) = \frac{(t+3)^2(t-2)}{(t+1)^4}.$$

If nothing divides the numerator or denominator at a given point, then the order of vanishing there is 0. In this case, we can split $f(t)$ into factors and read off the order of vanishing at each point. This leads to the definition of $\operatorname{div}(f)$.

[5.1.0.4] PROPOSITION. For any $f \in \mathbb{C}(X) \setminus \{0\}$, there are only finitely many prime divisors C such that

$$\nu_C(f) \neq 0.$$

[5.1.0.5] DEFINITION (*Divisor of f*).

$$\operatorname{div}(f) := \sum_C \nu_C(f)\, C,$$

where the sum runs over all prime divisors C.

[5.1.0.6] DEFINITION (*Principal Divisor*). A divisor $D \in \operatorname{Div}(X)$ is principal if there exists $f \in \mathbb{C}(X) \setminus \{0\}$ such that

$$D = \operatorname{div}(f).$$

For the f above, we obtain

$$\operatorname{div}(f) = 2\{t = -3\} + \{t = 2\} - 4\{t = -1\}.$$

In fact, any point on $\mathbb{A}^1$ gives a principal divisor, but this is not true on $\mathbb{P}^1$ unless we also account for the point at infinity. We find this intuition by thinking of projective space as

$$\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_u,$$

where $\infty = \{u = 0\} \subset \mathbb{A}^1_u$. For example,

$$\operatorname{div}(t) = \{t = 0\} - \{\infty\}.$$

Equivalently,

$$\nu_{\{\infty\}}(t) = -1.$$

5.2 Divisor Class Group

[5.2.0.1] DEFINITION (*Divisor Class Group*).

$$\begin{aligned} (X) &:= \operatorname{Div}(X)/\{\text{principal divisors}\} \\ &= \operatorname{Div}(X)/\sim, \end{aligned}$$

where $\sim$ denotes linear equivalence.

If $D, D' \in \operatorname{Div}(X)$ and $D - D'$ is principal, then we say D and D' are linearly equivalent, written $D \sim D'$. For example, $(\mathbb{A}^n) = 0$, since every divisor is principal. Indeed, if $D = \sum k_i C_i \subset \mathbb{A}^n$, where $C_i = \{H_i = 0\}$ and H_i is a polynomial, then

$$f = \prod_i H_i^{k_i}$$

satisfies $D = \operatorname{div}(f)$. However, $(\mathbb{P}^n) \cong \mathbb{Z}$. If $D = \sum k_i H_i$, then any $f \in \mathbb{C}(\mathbb{P}^n)$ must have degree 0, hence

$$\deg(D) = 0 \quad \text{for principal divisors.}$$

[5.2.0.2] Proposition. There is a natural degree map for X quasi-projective:

$$\begin{aligned} \deg : \operatorname{Div}(X) &\longrightarrow \mathbb{Z} \\ D = \sum k_i H_i &\longmapsto \sum k_i \deg(H_i). \end{aligned}$$

If $\deg(D) \neq 0$, then D is not principal.

The class group of a quasi-projective variety need not be finitely generated. Let X be a smooth projective curve of genus $g > 0$. If $X \ncong \mathbb{P}^1$, then there exist points $P, Q \in X$ such that $P \nsim Q$. Suppose for contradiction that $P \sim Q$. Then there exists $f \in \mathbb{C}(X)$ such that

$$\operatorname{div}(f) = P - Q.$$

This defines a rational map

$$f : X \dashrightarrow \mathbb{P}^1.$$

Since $\operatorname{div}(f)$ has a simple zero at P and a simple pole at Q, we have $\deg(f) = 1$. Thus f is birational, implying $X \cong \mathbb{P}^1$, a contradiction. Therefore, distinct points need not be linearly equivalent. Since there are uncountably many points on X, it follows that (X) is not finitely generated.

5.3 Local Divisors

[5.3.0.1] Definition (*Cartier Divisor*). A Cartier divisor is a locally principal Weil divisor.

Let $X = \bigcup U_i$ be an affine open cover. A divisor $D \in \operatorname{Div}(X)$ is called locally principal if $D|_{U_i}$ is principal for each i. If $X = \mathbb{P}^1$ and $D = (0)$, then D is not principal globally, but it is locally principal. Indeed, take the affine chart $\mathbb{A}^1_t = \{t \neq \infty\}$. Then $\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_u$ with $t = \dfrac{1}{u}$. The divisor $D = \{t = 0\}$ is principal on $\mathbb{A}^1_t$, and $D \cap \mathbb{A}^1_u = \varnothing$. Thus D is locally principal. The group of Cartier divisors is denoted (X). We define

$$(X) := (X)/\{\text{principal divisors}\}.$$

This group is important since

$$(X) \cong \operatorname{Pic}(X),$$

the Picard group of X, which consists of line bundles up to isomorphism.

[5.3.0.2] Theorem. If X is smooth, then every Weil divisor is Cartier.

An example of a Weil divisor that is not Cartier is

$$X = \{xy = z^2\} \subseteq \mathbb{C}^3.$$

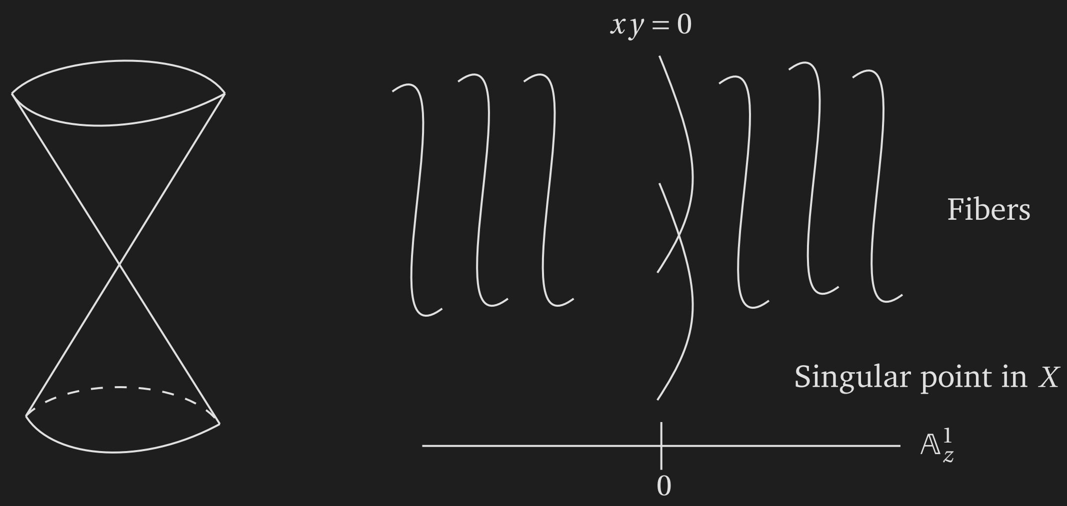


Figure 5.3: An affine surface with a singular point.

We have

$$f^{-1}(0) = \{xy = 0\} = \{x = 0\} \cup \{y = 0\}.$$

Let $D = \{y = 0\} \subseteq f^{-1}(0)$. We claim that D is not Cartier. Observe that

$$xy = z^2 \quad \Rightarrow \quad y = \frac{z^2}{x}.$$

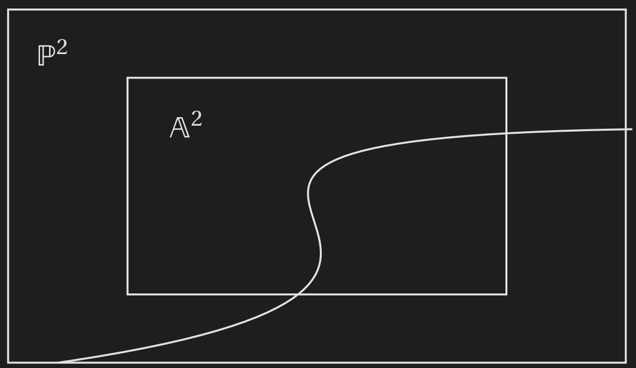
Thus along D, we have $\{y = 0\} = \{z = 0\}$. The function y vanishes to order 2 along D, so

$$\nu_D(y) = 2.$$

Hence

$$\mathrm{div}(y) = 2D.$$

This shows D is not principal, and in fact not Cartier. When defining $\mathrm{div}(f) = \sum \nu_C(f)\, C$, we choose $U \subset X$ such that $C \cap U = \{\pi = 0\}$.



Take π such that the ideal of functions vanishing on D is generated by π.

5.3.1 Q-Factorial

[5.3.1.1] DEFINITION (*Q-Factorial*). A quasi-projective variety X is Q-factorial if for every Weil divisor D, there exists $n \in \mathbb{N}$ such that nD is Cartier.

Recall in the example $X = \{xy = z^2\}$, we have $\operatorname{div}(y) = 2D$, which is Cartier. If X is an affine ***toric variety***, then X is Q-factorial if and only if its fan is ***simplicial***.

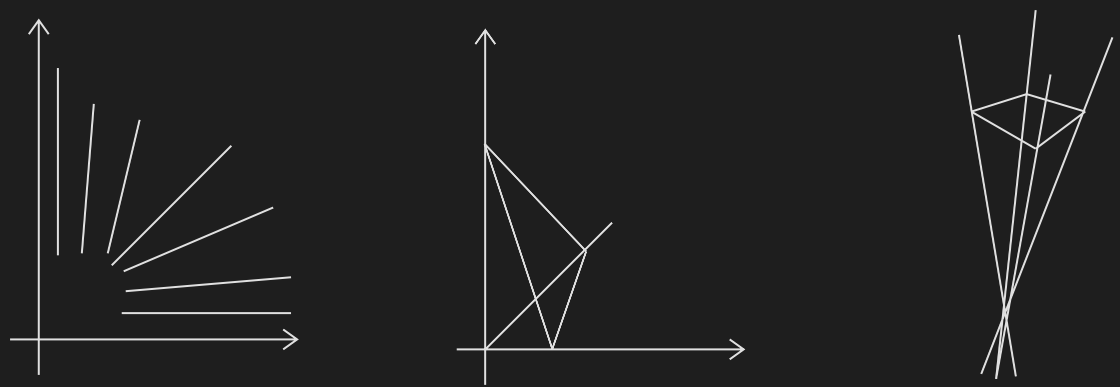


Figure 5.4: From left to right: $\mathbb{A}^2$, $\mathbb{A}^3$, $\mathbb{A}^4$: a conifold singularity which is not Q-factorial.

5.4 Linear Systems of Divisors

Let X be a nonsingular variety. All Weil divisors are Cartier on nonsingular varieties. If we have $D \subseteq X$ a divisor, there is an associated vector space $\mathcal{L}(D)$ defined by

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) \setminus \{0\} : \operatorname{div}(f) + D \geq 0\}.$$

If

$$\operatorname{div}(f) + D = \sum a_i C_i,$$

then $a_i \geq 0$ for all i. Thus $\operatorname{div}(f) + D$ is effective. If $D = \sum n_i C_i$, then $f \in \mathcal{L}(D)$ if and only if $\nu_{C_i}(f) \geq -n_i$ for each i, and $\nu_C(f) \geq 0$ for any $C \neq C_i$. Since D is Cartier, there exists $\mathcal{O}(D)$, a line bundle, and

$$\mathcal{L}(D) = H^0(X, \mathcal{O}(D)),$$

where H^0 denotes the space of sections of $\mathcal{O}(D)$. If $X = \mathbb{P}^1$ and $D = nX_\infty$, where the point at infinity is $[1:0]$, then we ask: what is $\mathcal{L}(D)$? Let $f \in \mathbb{C}(\mathbb{P}^1) \setminus \{0\}$. Then $f = p/q$, where p, q are homogeneous polynomials of the same degree, say d. Let $\{p_1, \ldots, p_d\}$ denote the zeros of $p(x_0, x_1)$, and similarly for q. Then

$$\operatorname{div}(f) = \sum p_i - \sum q_i.$$

We want $\operatorname{div}(f) + D \geq 0$, where $D = nX_\infty$. If there exists some zero $q_i \neq X_\infty$, then this is not true. Otherwise,

$$\begin{aligned}\operatorname{div}(f) + D &= \sum p_i - dX_\infty + nX_\infty \\ &= \sum p_i + (n-d)X_\infty.\end{aligned}$$

This is effective if and only if $n-d \geq 0$. Then $q = cX_1^d$, and we have

$$f = \frac{\sum_{i=0}^d a_i X_0^i X_1^{d-i}}{cX_1^d}.$$

If $a_i' = a_i/c$, then

$$f = \sum_{i=0}^d a_i' \left(\frac{x_0}{x_1}\right)^i.$$

So $f \in \mathbb{C}[x_0/x_1]_{\leq n}$, meaning the highest degree is at most n. Hence $\mathcal{L}(D)$ is a $\mathbb{C}$-vector space of dimension $n+1$. If $n \leq -1$, then $\mathcal{L}(D) = \{0\}$.

5.4.1 Hypersurfaces

[5.4.1.1] THEOREM. If X is a smooth projective variety, then $\mathcal{L}(D)$ is finite-dimensional.

We write $\dim_{\mathbb{C}}(\mathcal{L}(D)) = \ell(D)$. Why do we need projectivity? Let D be the zero divisor. Then

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) \setminus \{0\} : \operatorname{div}(f) \geq 0\}.$$

So f has no poles, hence f is regular. If $X = \mathbb{P}^1$, then $\mathcal{L}(D) = \mathbb{C}[\mathbb{P}^1] = \mathbb{C}$. If $X = \mathbb{A}^1$, then $\mathcal{L}(D) = \mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t]$. Then $\ell(D) = \infty$.

[5.4.1.2] DEFINITION (*Linear System*). Let X be projective. Then the linear system of $D \in \operatorname{Div}(X)$ is the projectivization of $\mathcal{L}(D)$, denoted $\mathbb{P}(\mathcal{L}(D))$. This is isomorphic to

$$\frac{\mathcal{L}(D) \setminus \{0\}}{\mathbb{C}^*} = \mathbb{P}^{\ell(D)-1}.$$

[5.4.1.3] PROPOSITION. Suppose X is projective. If $D \sim D'$, then $\mathcal{L}(D) \cong \mathcal{L}(D')$.

∴

Proof. If $D \sim D'$, then $D - D' = \operatorname{div}(g)$ for some $g \in \mathbb{C}(X)$. Consider

$$\varphi : \mathcal{L}(D) \to \mathcal{L}(D')$$

defined by

$$\varphi(f) = fg.$$

If $f \in \mathcal{L}(D)$, then $\operatorname{div}(f) + D \geq 0$. Now

$$\begin{aligned} \operatorname{div}(fg) + D' &= \operatorname{div}(f) + \operatorname{div}(g) + D' \\ &= \operatorname{div}(f) + (D - D') + D' \\ &= \operatorname{div}(f) + D \geq 0. \end{aligned}$$

So $fg \in \mathcal{L}(D')$. Conversely, the inverse map

$$\varphi^{-1} : \mathcal{L}(D') \to \mathcal{L}(D)$$

is defined by

$$h \mapsto \frac{h}{g}.$$

Thus $\mathcal{L}(D) \cong \mathcal{L}(D')$. □

In particular, $\mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}(\mathcal{L}(D'))$. One can check that $\mathbb{P}(\mathcal{L}(D))$ is naturally identified with the set of effective divisors linearly equivalent to D. If $f \in \mathcal{L}(D) \setminus \{0\}$, then

$$\operatorname{div}(f) + D \geq 0$$

is an effective divisor linearly equivalent to D. Moreover, if $c \in \mathbb{C}^*$, then $\operatorname{div}(cf) = \operatorname{div}(f)$, so the associated effective divisor is unchanged. Let $X = \mathbb{P}^n$ and let $D = \{f_d = 0\}$ be a hypersurface of degree d. Let D' be defined similarly. Then

$$D - D' = \operatorname{div}\left(\frac{f_d}{f'_d}\right),$$

since $\deg(f_d) = \deg(f'_d)$. Hence

$$\mathcal{L}(D) \cong \{\text{degree } d \text{ homogeneous polynomials in } x_0, \ldots, x_n\}.$$

Also,

$$\frac{f_d}{f'_d} \in \mathbb{C}(\mathbb{P}^n).$$

We already know how many degree d homogeneous polynomials there are, by stars and bars. Therefore

$$\mathbb{P}(\mathcal{L}(D)) = \mathbb{P}^{\binom{n+d}{d}-1},$$

which is the projective space of all hypersurfaces of degree d in $\mathbb{P}^n$. Let D be a cubic curve in $\mathbb{P}^2$. Then

$$\mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}^9.$$

We will later show that for a genus g curve, the divisor class group is not countable. With Riemann–Roch, we know $Cl(\mathbb{P}^n) = \mathbb{Z}$, while $Cl\left(\sum p\right)$ is not countable. This is not subtle.

5.5 Degree of a Divisor

Let X be a smooth projective curve. Let

$$D = \sum a_i p_i \in \operatorname{Div}(X).$$

[5.5.0.1] Definition (*Degree of a Divisor*). The degree of D is

$$\deg(D) := \sum a_i.$$

Let $X = \mathbb{P}^1$. A divisor D is principal if and only if $\deg(D) = 0$. Moreover,

$$Cl(X) = \frac{\operatorname{Div}(X)}{\{\text{principal divisors}\}} \cong \mathbb{Z},$$

via the map
$$[D] \mapsto \deg(D).$$

[5.5.0.2] THEOREM. Let X be a smooth projective curve. If D is principal, then $\deg(D) = 0$.

Proof. Suppose D is principal. Then
$$D = \operatorname{div}(f) = \sum \nu_{p_i}(f)\, p_i$$
for some $f \in \mathbb{C}(X)$. Thus
$$D = \sum Z(f) - \sum P(f),$$
where $Z(f)$ and $P(f)$ denote the zeros and poles of f respectively.

It suffices to show that
$$\deg(Z(f)) = \deg(P(f)).$$

If f is nonconstant, then it defines a rational map
$$f : X \dashrightarrow \mathbb{C}.$$
This extends to a morphism
$$f : X \to \mathbb{P}^1.$$
This map is finite, hence a ramified covering. Define $\deg(f)$ as the number of preimages of a general point.

We claim that $\deg(f)$ equals the number of preimages of any point in $\mathbb{P}^1$, counted with multiplicity. Indeed, for any $p \in \mathbb{P}^1$ and $y \in f^{-1}(p)$, there exists a neighborhood $U \subset X$ such that
$$f|_U : U \to \mathbb{C}$$
is locally of the form $z \mapsto z^k$ for some $k \in \mathbb{N}$. Thus each point contributes with multiplicity.

Therefore, counting multiplicities,
$$\deg(Z(f)) = \deg(P(f)),$$
so $\deg(D) = 0$. ■

The degree is invariant under linear equivalence. Hence there is a well-defined group homomorphism
$$\deg : Cl(X) \to \mathbb{Z}.$$
Define
$$Cl^0(X) := \ker(\deg).$$
Then we have a short exact sequence
$$0 \to Cl^0(X) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0,$$
so
$$Cl(X) \cong \mathbb{Z} \oplus Cl^0(X).$$

[5.5.0.3] THEOREM. If $X \neq \mathbb{P}^1$, then $Cl^0(X) \neq 0$. Moreover, it is uncountable.

In fact,

$$Cl^0(X) \cong J(X),$$

the Jacobian of X. If X has genus g, then $J(X)$ is a g-dimensional abelian variety. As a complex manifold,

$$J(X) \cong \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}.$$

5.6 Bézout's Theorem

[5.6.0.1] DEFINITION (*Hypersurface*). A hypersurface $H \subseteq \mathbb{P}^n$ is a projective variety of the form

$$H = \{F = 0\},$$

where $F \in \mathbb{C}[x_0, \dots, x_n]$ is homogeneous.

[5.6.0.2] THEOREM. Let $X \subset \mathbb{P}^n$ be a smooth projective curve. Let $H \subset \mathbb{P}^n$ be a hypersurface not containing X. Then

$$X \cdot H = \deg(X)\deg(H).$$

If $H = \{F = 0\}$, then $\deg(H) = \deg(F) = d$.

[5.6.0.3] DEFINITION (*Hyperplane*). A hyperplane $H' \subset \mathbb{P}^n$ is a projective linear subspace of codimension 1. If H' intersects X transversely, then

$$\deg(X) = \#(X \cap H').$$

[5.6.0.4] DEFINITION. We say X and H' meet transversely at $P \in X \cap H'$ if

$$T_P(X) + T_P(H') = T_P(\mathbb{P}^n).$$

Construction of the Intersection Divisor

Since F is not a global function on $\mathbb{P}^n$, we work locally.

(1) Assume $X \not\subset \{x_i = 0\}$ for some i. Otherwise, perform a projective change of coordinates.

(2) Let

$$U_i = \{x_i \neq 0\}, \quad \mathbb{P}^n = \bigcup_{i=0}^{n} U_i.$$

Define

$$f_i = \frac{F}{x_i^d}\bigg|_{X \cap U_i}.$$

Then f_i is regular on $X \cap U_i$.

(3) The divisor $\mathrm{div}(f_i)$ on $X \cap U_i$ is effective and supported on

$$X \cap U_i \cap H.$$

These local divisors glue to give a global divisor $\mathrm{div}(F)$ on X.

> **[5.6.0.5]** Definition (*Intersection Number*). The intersection number of X and H is
> $$X \cdot H := \deg(\mathrm{div}(F)).$$

Proof of Bézout's Theorem

Proof. Let $H = \{F = 0\}$ and $H' = \{F' = 0\}$ be hypersurfaces with $\deg(F) = \deg(F')$. Then $H \sim H'$.

We claim

$$X \cdot H = X \cdot H'.$$

Observe

$$\mathrm{div}(F) - \mathrm{div}(F') = \mathrm{div}\left(\frac{F}{F'}\right).$$

Since $\deg(F) = \deg(F')$, we have

$$\frac{F}{F'} \in \mathbb{C}(\mathbb{P}^n), \qquad \frac{F}{F'}\bigg|_X \in \mathbb{C}(X).$$

Thus $\mathrm{div}(F) \sim \mathrm{div}(F')$ on X.

On a smooth projective curve, linearly equivalent divisors have the same degree. Hence

$$X \cdot H = X \cdot H'.$$

Now factor F' into linear forms:

$$F' = L_1 \cdots L_d.$$

Then

$$H' = H_1 \cup \cdots \cup H_d, \quad H_i = \{L_i = 0\}.$$

Therefore

$$X \cdot H' = \sum_{i=1}^{d} X \cdot H_i.$$

Each H_i is a hyperplane, so

$$X \cdot H_i = \deg(X).$$

Thus

$$X \cdot H' = d \cdot \deg(X) = \deg(H)\deg(X).$$

Hence

$$X \cdot H = \deg(X)\deg(H).$$

■

Examples

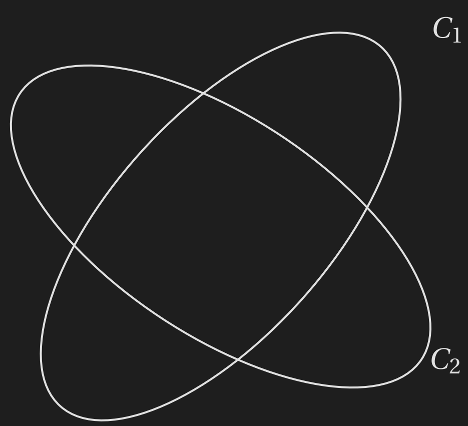


Figure 5.5: Two curves intersecting in the expected number of points.

In general position, the number of intersection points equals $\deg(C_1)\deg(C_2)$.

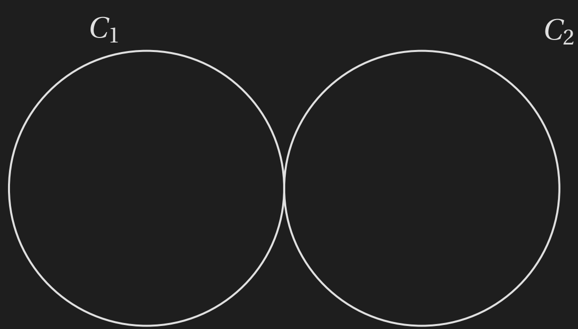


Figure 5.6: Affine intersection vs. projective intersection.

Let $\mathbb{P}^2_{[x,y,z]}$ with affine chart $z \neq 0$. Then

$$C_1 : x^2 + y^2 = z^2, \quad C_2 : (x-z)^2 + y^2 = z^2.$$

On the affine chart $z = 1$:

$$x^2 + y^2 = 1, \quad (x-1)^2 + y^2 = 1.$$

Subtracting gives

$$x^2 = x^2 - 2x + 1 \quad \Rightarrow \quad x = \frac{1}{2}.$$

Thus

$$y = \pm\frac{\sqrt{3}}{2}.$$

There are 2 affine intersection points, but Bézout predicts 4. The remaining intersection points lie at infinity ($z = 0$), which are invisible in the affine chart.

5.7 Dimension of Divisors

Let X be a smooth projective variety and let D be a divisor on X. Define

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + D \geq 0\}.$$

Then $\mathcal{L}(D)$ is a $\mathbb{C}$-vector space of finite dimension.

[5.7.0.1] Definition (*Dimension of a Divisor*). The dimension of a divisor D is

$$\ell(D) := \dim_{\mathbb{C}} \mathcal{L}(D).$$

Basic Bound

[5.7.0.2] Theorem. Let X be a smooth projective curve and D a divisor on X. If D is effective or a 0-divisor, then

$$\ell(D) \leq \deg(D) + 1.$$

If $D = \sum a_i p_i$, then $\deg(D) = \sum a_i$. In particular, this theorem implies $\ell(D)$ is finite. If $D = 0$, then

$$\mathcal{L}(D) = \mathbb{C}[X] = \mathbb{C},$$

so $\ell(D) = 1$. Since $\deg(D) = 0$, we obtain equality:

$$\ell(D) = \deg(D) + 1.$$

Case $X = \mathbb{P}^1$

If $D \sim D'$, then $\mathcal{L}(D) \cong \mathcal{L}(D')$. Thus we may assume

$$D = \deg(D) \cdot p_\alpha.$$

Then

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(\mathbb{P}^1) : \operatorname{div}(f) + D \geq 0\}.$$

This means f is regular on $\mathbb{P}^1 \setminus \{p_\alpha\}$ and has a pole of order at most $\deg(D)$ at p_α. Choose an affine chart

$$\mathbb{P}^1 \setminus \{p_\alpha\} \cong \mathbb{A}^1_t.$$

Then $f \in \mathbb{C}[t]$. Using the coordinate $u = \frac{1}{t}$ near p_α, the condition becomes

$$\deg(f) \leq \deg(D).$$

Hence

$$\mathcal{L}(D) \cong \{f \in \mathbb{C}[t] : \deg(f) \leq \deg(D)\} \cong \mathbb{C}^{\deg(D)+1}.$$

Therefore

$$\ell(D) = \deg(D) + 1.$$

Nontrivial Case: $X \neq \mathbb{P}^1$

Let X be a smooth projective curve not isomorphic to $\mathbb{P}^1$, and let

$$D = p \in X.$$

Then

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + p \geq 0\}.$$

Thus f is regular on $X \setminus \{p\}$ and has at most a first-order pole at p.

Proof. If f has no poles, then $f \in \mathbb{C}[X]$, hence f is constant.

Assume f has a first-order pole at p. Then f defines a morphism

$$f : X \to \mathbb{P}^1.$$

Since the pole has order 1, we have $\deg(f) = 1$. Thus f is an isomorphism, implying $X \cong \mathbb{P}^1$, a contradiction.

Therefore f must be constant. Hence

$$\mathcal{L}(D) \cong \mathbb{C}, \quad \ell(D) = 1.$$

Since $\deg(D) = 1$, we obtain

$$\ell(D) = 1 < \deg(D) + 1 = 2.$$

∎

This shows that equality $\ell(D) = \deg(D) + 1$ does not hold in general.

Chapter 6

Differential Forms

6.1 Regular Differential 1-Forms

Let X be a quasi-projective variety. We define regular differential 1-forms on X. Let f be a regular function on a neighbourhood U of a point $x \in X$, so $f \in \mathbb{C}[U]$. The differential of f at x is denoted

$$d_x f \in T_x^*,$$

the cotangent space at x. If $X = \mathbb{A}^n$ with coordinates $t_1, \ldots, t_n$, then $T_x \cong \mathbb{C}^n$. For $f = t_i$, we have

$$d_x t_i : \mathbb{C}^n \to \mathbb{C},$$

which is the projection onto the i-th coordinate. More generally, for $f \in \mathbb{C}[t_1, \ldots, t_n]$,

$$d_x f : \mathbb{C}^n \to \mathbb{C}, \quad (u_1, \ldots, u_n) \mapsto \sum_{i=1}^n \left.\frac{\partial f}{\partial t_i}\right|_x u_i.$$

If X is not affine, we work locally on affine neighbourhoods. Define

$$\Phi(X) = \left\{ \varphi : X \to \coprod_{x \in X} T_x^* \;\middle|\; \varphi(x) \in T_x^* \right\}.$$

[6.1.0.1] Definition (*Regular Differential 1-Form*). A regular 1-form on X is a function $\varphi \in \Phi(X)$ such that for every $x \in X$, there exists a neighbourhood U of x with

$$\varphi|_U = \sum_i f_i \, dg_i,$$

where $f_i, g_i \in \mathbb{C}[U]$.

Equivalently, using sheaf language, a regular 1-form is a global section of the cotangent sheaf. Explicitly,

$$\sum_i f_i \, dg_i : U \to \coprod_{x \in U} T_x^*, \quad x \mapsto \sum_i f_i(x) \, d_x g_i.$$

[6.1.0.2] REMARK. Let $\Omega[X]$ denote the set of all regular 1-forms on X. Then $\Omega[X]$ is a $\mathbb{C}[X]$-module.

We have the natural differential map

$$d : \mathbb{C}[X] \to \Omega[X]$$
$$f \mapsto (x \mapsto d_x f),$$

which satisfies

$$d(f+g) = df + dg, \quad d(fg) = f\,dg + g\,df.$$

[6.1.0.3] THEOREM. If X is projective, then $\Omega[X]$ is a finite-dimensional $\mathbb{C}$-vector space.

Since X is projective, $\mathbb{C}[X] = \mathbb{C}$, so $\Omega[X]$ is naturally a $\mathbb{C}$-vector space. Define

$$\dim_{\mathbb{C}} \Omega[X] := h^{1,0}.$$

If X is a smooth projective curve, then $h^{1,0}$ is the geometric genus of X.

6.2 Rational Differential 1-Forms

Let X be a quasi-projective variety. A function $f \in \mathbb{C}(X)$ is rational if it is regular on some open dense subset $U \subseteq X$, i.e. $f \in \mathbb{C}[U]$. Analogously, define $\Omega(X)$ to be the space of all rational differential 1-forms on X. The choice of U does not matter: if $\omega \in \Omega(U)$ and $\omega' \in \Omega(U')$ agree on $U \cap U'$, then they define the same rational 1-form on X. The space $\Omega(X)$ is a $\mathbb{C}(X)$-vector space. Indeed, for $\omega, \omega' \in \Omega(X)$ and $f \in \mathbb{C}(X)$,

$$\omega + \omega' \in \Omega(X), \quad f\omega \in \Omega(X).$$

Dimension

We claim

$$\dim_{\mathbb{C}(X)} \Omega(X) = \dim(X).$$

Affine Case

Let $X = \mathbb{A}^n$ with coordinates $t_1, \dots, t_n$. A rational 1-form has the form

$$\sum_i f_i\,dg_i, \quad f_i, g_i \in \mathbb{C}(X).$$

Then

$$\{dt_1, \dots, dt_n\}$$

forms a basis for $\Omega(X)$ over $\mathbb{C}(X)$. Hence

$$\dim_{\mathbb{C}(X)} \Omega(X) = n = \dim(X).$$

Projective Case

If $X = \mathbb{P}^n$, then

$$\Omega[\mathbb{P}^n] = 0,$$

since all regular functions are constant. However,

$$\Omega(\mathbb{P}^n)$$

is an n-dimensional vector space over $\mathbb{C}(\mathbb{P}^n)$. Moreover,

$$\Omega(\mathbb{P}^n) \cong \Omega(\mathbb{A}^n)$$

on affine charts.

Birational Invariance

In general, if X and X' are birational varieties, then there is a natural identification

$$\Omega(X) \cong \Omega(X').$$

6.3 Behavior Under Maps of Differential 1-Forms

Let X, Y be quasi-projective varieties and let $\varphi : X \to Y$ be a regular map. There exists a pullback map on 1-forms

$$\varphi^{*:\Omega[Y]\to\Omega[X].}$$

Example

Let $\varphi : \mathbb{A}^1_t \to \mathbb{A}^1_u$ be defined by $t \mapsto u = t^k$. Then

$$\varphi^{*:\Omega[\mathbb{A}^1_u]\to\Omega[\mathbb{A}^1_t]}$$

is given by

$$f(u)\,du \longmapsto f(t^k)\,d(t^k), \quad d(t^k) = kt^{k-1}\,dt.$$

[6.3.0.1] THEOREM. If X, Y are smooth projective varieties and birational, then

$$\Omega[X] \cong \Omega[Y].$$

If X is birational to Y, there exists a rational map $\varphi : X \dashrightarrow Y$. Then φ is regular on some open subset $U \subseteq X$. For $\omega \in \Omega[Y]$, the pullback

$$(\varphi|_U)^*\omega$$

is a regular 1-form on U, hence a rational 1-form on X.

[6.3.0.2] THEOREM. If φ is birational, then $(\varphi|_U)^*\omega$ extends to a regular 1-form on X.

Thus $\varphi^{*:\Omega[Y]\to\Omega[X]}$ is well-defined even when φ is only rational, provided X, Y are smooth projective varieties. Consequently,

$$\dim \Omega[X] = \dim \Omega[Y],$$

so these dimensions (the Hodge numbers $h^{1,0}$) are birational invariants.

Canonical Divisor

Let $\omega, \omega' \in \Omega(X)$ be nonzero rational 1-forms. Since $\Omega(X)$ is a 1-dimensional vector space over $\mathbb{C}(X)$ (when X is a curve), we have

$$\omega' = g\omega$$

for some $g \in \mathbb{C}(X)$. Thus

$$\operatorname{div}(\omega') = \operatorname{div}(g) + \operatorname{div}(\omega).$$

Since $\operatorname{div}(g)$ is principal, we obtain

$$\operatorname{div}(\omega) \sim \operatorname{div}(\omega').$$

Hence $\operatorname{div}(\omega)$ defines a well-defined class in $Cl(X)$, called the *canonical class* K_X.

Relation with $\mathcal{L}(K_X)$

For a divisor D, recall

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + D \geq 0\}.$$

Then

$$\mathcal{L}(K_X) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + K_X \geq 0\}.$$

Since $K_X = \operatorname{div}(\omega)$,

$$\operatorname{div}(f) + \operatorname{div}(\omega) \geq 0 \quad \iff \quad \operatorname{div}(f\omega) \geq 0.$$

Thus $f\omega$ is a regular 1-form, so

$$\mathcal{L}(K_X) \cong \Omega[X].$$

Example: $X = \mathbb{P}^1$

Let $\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_u$ with $t = \frac{1}{u}$. Take $\omega = dt$, which is a rational 1-form on $\mathbb{P}^1$. On $\mathbb{A}^1_t$, we have

$$\operatorname{div}(\omega)|_{\mathbb{A}^1_t} = 0.$$

On $\mathbb{A}^1_u$, since

$$t = \frac{1}{u}, \quad dt = -\frac{1}{u^2}du,$$

we obtain

$$\operatorname{div}(\omega)|_{\mathbb{A}^1_u} = \operatorname{div}(-u^{-2}) = -2\{u = 0\}.$$

Hence

$$\deg(K_X) = -2.$$

Problem

Show that the rational differential form $\frac{dx}{y}$ is regular on the affine curve

$$X = \{x^2 + y^2 = 1\}.$$

If $y \neq 0$, then $\frac{dx}{y}$ is clearly regular. If $y = 0$, this occurs at two points p_1, p_2. At these points, tangent vectors are vertical, so $dx = 0$ on all tangent directions. Thus $\frac{dx}{y}$ appears indeterminate. Using the relation

$$x^2 + y^2 = 1,$$

differentiate to obtain

$$2x\,dx + 2y\,dy = 0, \quad \Rightarrow \quad \frac{dx}{y} = -\frac{dy}{x}.$$

This expression is regular at p_1, p_2. Hence $\frac{dx}{y}$ is regular everywhere on X.

Structure of $\Omega^1[X]$

We claim

$$\Omega^1[X] = \mathbb{C}[X]\frac{dx}{y}.$$

First, $\frac{dx}{y}$ is regular, and $\Omega^1[X]$ is a $\mathbb{C}[X]$-module, so

$$\mathbb{C}[X]\frac{dx}{y} \subseteq \Omega^1[X].$$

Conversely, let $\omega \in \Omega^1[X]$. Then $\omega \in \Omega(X)$, and since $\dim \Omega(X) = 1$, we can write

$$\omega = f\frac{dx}{y}, \quad f \in \mathbb{C}(X).$$

Since ω is regular and $\frac{dx}{y}$ has no poles, we must have $f \in \mathbb{C}[X]$. Thus

$$\Omega^1[X] \subseteq \mathbb{C}[X]\frac{dx}{y}.$$

Therefore,

$$\Omega^1[X] = \mathbb{C}[X]\frac{dx}{y}.$$

6.4 Hypersurfaces

Let $X \subset \mathbb{P}^2$. In this section we want to describe the space of regular 1-forms $\Omega[X]$, when X is a smooth curve in $\mathbb{P}^2$, and calculate the dimension of $\Omega[X]$ and the canonical class K_x. Let $X = \{F(x_0, x_1, x_2) = 0\}$ a curve of homogenous polynomial of degree d. Assume that X is smooth. The only solution to $\frac{\partial F}{\partial x} = 0$ for all i and $F = 0$ is the point $[0:0:0] \notin \mathbb{P}^2$. $F(x_0, x_1, x_2) = x_0^4 + x_1^4 + x_2^4$ defines a degree 4 smooth curve in $\mathbb{P}^2$. Let $U \cong \mathbb{A}^2$ be the affine chart where $x_0 \neq 0$, implying affine coordinates on U are $y_1 = \frac{x_1}{x_0}$ and $y_2 = \frac{x_2}{x_0}$. Denote $G \in \mathbb{C}[y_1, y_2]$ as a polynomial in y_1, y_2 such that $G(y_1, y_2) = F(1, y_1, y_2$. We can then define $G = 1 + y_1^4 + y_2^4$ so $\{G = 0\}$ defines an affine curve $X \cap U$. If $\frac{\partial G}{\partial y_1} \neq 0$, then ω is regular. Assume $\frac{\partial G}{\partial y_1} = 0$: first note that we can write

$$\omega = -\frac{1}{\frac{\partial G}{\partial y_1}}dy_2 = \frac{1}{\frac{\partial G}{\partial y_2}}dy_1$$

Equivalently,

$$dG = \frac{\partial G}{\partial y_1}dy_1 + \frac{\partial G}{\partial y_2}dy_2 = 0$$

This equation is satisfied on the curve $X \cap U$, because this equation is the equation of the tangent space to the curve. That is $G = 0$ implies $dg = 0$. Since X is smooth, we cannot have both $\frac{\partial G}{\partial y_1} = 0$ and $\frac{\partial G}{\partial y_2} = 0$. So, if $\frac{\partial G}{\partial y_1} = 0$, we must have $\frac{\partial G}{\partial y_2} \neq 0$. In this case, $\omega = \frac{1}{\frac{\partial G}{\partial y_2}}dy_1$ is regular. Therefore ω is a nowhere zero regular 1-form $\mathrm{div}(\omega) = 0$ on $X \cap U$. That is ω has no zeroes and no poles, meaning $\omega = \frac{1}{\frac{\partial G}{\partial y_2}}dy_1$ and is regular respectively.

Consider the other affine chart $V = \{x_1 \neq 0\}$ with coordinates $z_1 = \frac{x_0}{x_1}$ and $z_2 = \frac{x_2}{x_1}$. Hence $X \cap V = \{H(z_1, z_2) = 0\}$.

We don't have to look at the third chart.

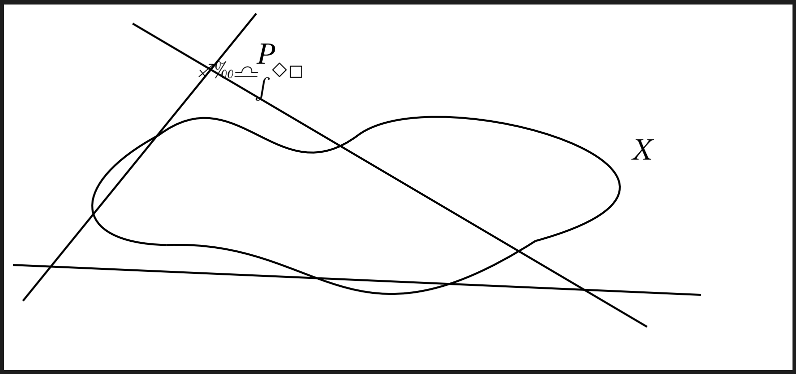


Figure 6.1: Because $\mathbb{P}^2 \setminus \{U \cup V\} = p$

We can always choose X to avoid this point. So, $\operatorname{div}(\omega) = 0$ on U and $\operatorname{div}(\omega) = (d-3)D$ on V. Thus $\operatorname{div}(\omega) = (d-3)D$ on X is a canonical divisor of class K_X.

The degree of K_X is $\deg(K_X) = d(d-3)$. If $d = 1$ meaning $X \cong \mathbb{P}^1$ line, then $\deg(K_x) = -2$. If $d = 2$ meaning $X \cong \mathbb{P}^1$ a smooth conic, then $\deg(K_X) = 2(-1) = -2$.

6.5 Riemann-Roch Theorem

Let X be a smooth projective curve. Let K_X denote the canonical divisor class. For a divisor D, define

$$\mathcal{L}(D) = \{0\} \cup \{f \in \mathbb{C}(X) : \operatorname{div}(f) + D \geq 0\},$$

and

$$\ell(D) = \dim(\mathcal{L}(D)).$$

Recall that $\mathcal{L}(K_X) \cong \Omega[X]$. Thus

$$\ell(K_X) = \dim(\Omega[X]) = g.$$

If $X \subset \mathbb{P}^2$ is a smooth curve of degree d, then

$$g = \frac{(d-1)(d-2)}{2}.$$

If $d = 3$, then $g = 1$. If $d = 4$, then $g = 3$. A smooth curve of genus 2 cannot lie in $\mathbb{P}^2$, but can lie in some $\mathbb{P}^n$.

[6.5.0.1] THEOREM. Any smooth projective curve can be embedded into $\mathbb{P}^3$.

The projectivization satisfies

$$\mathbb{P}(\mathcal{L}(D)) = \{D' \geq 0 : D' \sim D\}.$$

[6.5.0.2] THEOREM. (Riemann–Roch) Let X be a smooth projective curve and D a divisor on X. Then

$$\ell(D) - \ell(K_X - D) = 1 - g + \deg(D).$$

The difference $\ell(D) - \ell(K_X - D)$ depends only on g and $\deg(D)$. However, $\ell(D)$ itself depends on the divisor. Let X be a curve with $g > 0$. Let $D_1 = 0$. Then $\deg(D_1) = 0$ and $\mathcal{L}(D_1) = \mathbb{C}[X] = \mathbb{C}$. Thus $\ell(D_1) = 1$. Let $D_2 = p - q$ for distinct points $p, q \in X$. Then $\deg(D_2) = 0$.

Proof. If $\ell(D_2) > 0$, then there exists $f \in \mathbb{C}(X)$ with a simple zero at p and a simple pole at q. Thus $f : X \to \mathbb{P}^1$ has degree 1, hence is an isomorphism. This contradicts $g > 0$. Therefore $\ell(D_2) = 0$. ∎

Thus $\deg(D_1) = \deg(D_2)$ but $\ell(D_1) \neq \ell(D_2)$. Applying Riemann–Roch:

$$\ell(D_1) - \ell(K_X) = 1 - g$$

implies

$$\ell(K_X) = g.$$

$$\ell(D_2) - \ell(K_X - D_2) = 1 - g$$

implies

$$\ell(K_X - D_2) = g - 1.$$

Taking $D = K_X$ in Riemann–Roch:

$$\ell(K_X) - \ell(0) = 1 - g + \deg(K_X).$$

Since $\ell(K_X) = g$ and $\ell(0) = 1$, we obtain

$$g - 1 = 1 - g + \deg(K_X).$$

Hence

$$\deg(K_X) = 2g - 2.$$

For plane curves:

$$\deg(K_X) = d(d-3).$$

Thus

$$d(d-3) = 2g - 2 \implies g = \frac{(d-1)(d-2)}{2}.$$

If $\deg(D) > 2g - 2$, then $\deg(K_X - D) < 0$. Thus $\ell(K_X - D) = 0$ and

$$\ell(D) = 1 - g + \deg(D).$$

[6.5.0.3] THEOREM. Let X be a smooth projective curve and $f \in \mathbb{C}(X)$ with $\omega \in \Omega[X]$. Then

$$\sum_{x \in X} \mathrm{Res}_x(f\omega) = 0.$$

Pick a local coordinate t around x such that

$$f\omega = \left(\frac{a_k}{t^k} + \cdots + \frac{a_1}{t} + a_0 + \cdots\right) dt.$$

Then

$$\mathrm{Res}_x(f\omega) = a_1.$$

Proof. By Cauchy's formula,

$$\mathrm{Res}_x(f\omega) = \frac{1}{2\pi i} \int f\omega.$$

Summing over poles $p_1, \ldots, p_n$:

$$\sum_{i=1}^{n} \mathrm{Res}_{p_i}(f\omega) = \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{\gamma_i} f\omega.$$

By Stokes' theorem,

$$= \frac{1}{2\pi i} \int_{X \setminus \bigcup D_i} d(f\omega).$$

Locally $f\omega = g(t)dt$, so

$$d(g(t)dt) = g'(t)\, dt \wedge dt = 0.$$

Thus the sum is 0. ∎

[6.5.0.4] THEOREM. If $g = 0$, then $X \cong \mathbb{P}^1$.

Proof. Let $D = p$. Then $\deg(D) = 1 > 2g - 2 = -2$. Thus

$$\ell(D) = 1 - 0 + 1 = 2.$$

So there exists a non-constant $f \in \mathbb{C}(X)$ with a pole of order at most 1. Thus f defines an isomorphism $X \cong \mathbb{P}^1$. ∎

[6.5.0.5] THEOREM. If $g = 1$, then $K_X \sim 0$.

Proof. We have $\deg(K_X) = 2g - 2 = 0$. Since $\ell(K_X) = g = 1$, there exists an effective divisor linearly equivalent to K_X. The only effective divisor of degree 0 is 0. Thus $K_X \sim 0$. ∎

[6.5.0.6] THEOREM. If $g = 1$, then X is isomorphic to a cubic curve in $\mathbb{P}^2$.

Proof. Fix $p \in X$. Then $\ell(p) = 1$, $\ell(2p) = 2$, and $\ell(3p) = 3$.

Thus there exist functions $x \in \mathcal{L}(2p)$ and $y \in \mathcal{L}(3p)$ with poles of order 2 and 3 at p.

The map $(x, y) : X \to \mathbb{P}^2$ is an embedding.

Consider $\mathcal{L}(6p)$. Its dimension is 6. The functions $1, x, x^2, x^3, y, y^2, xy$ lie in $\mathcal{L}(6p)$ and are linearly dependent.

This gives a cubic relation in x, y, so X is a cubic curve. ∎

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